COHOMOLOGY OF STANDARD MODULES ON PARTIAL FLAG VARIETIES

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Abstract. Cohomological induction gives an algebraic method for constructing representations of a real reductive Lie group G from irreducible representations of reductive subgroups. Beilinson-Bernstein localization alternatively gives a geometric method for constructing Harish-Chandra modules for G from certain representations of a Cartan subgroup. The duality theorem of Hecht, Miličić, Schmid and Wolf establishes a relationship between modules cohomologically induced from minimal parabolics and the cohomology of the $\mathscr{D}$-modules on the complex flag variety for G determined by the Beilinson-Bernstein construction. The main results of this paper give a generalization of the duality theorem to partial flag varieties, which recovers cohomologically induced modules arising from nonminimal parabolics.

1. Introduction

The objective of this paper is to extend the duality theorem of [5] to partial flag varieties. For $G_{\mathbb{R}}$ be a real reductive Lie group and $(\mathfrak{g}, K)$ its complex Harish-Chandra pair, the main difference between the geometry of K-orbits on the full flag variety of $\mathfrak{g}$ and K-orbits on partial flag varieties is that the orbits are not necessarily affinely embedded in the case of partial flag varieties, whereas they are for the full flag variety. The affineness of the embedding of K-orbits in the full flag variety of $\mathfrak{g}$ was used in an essential way in [5]. Motivated by the derived equivariant constructions of [13] and [12], we define analogous geometric constructions which allow us to prove our main result using derived category techniques to take into account the failure of affineness of K-orbit embeddings.

1.1. **Main Theorem.** Before stating our main result, we first recall the duality theorem of [5]. As above, let $G_{\mathbb{R}}$ denote a real reductive Lie group, to which we associate its complex Harish-Chandra pair $(\mathfrak{g}, K)$ and abstract Cartan triple $(\mathfrak{h}, \Sigma, \Sigma^+)$. On the full flag variety X of $\mathfrak{g}$, let Q be a K-orbit and τ an irreducible connection on Q. There is a twisted sheaf of differential operators $\mathscr{D}_\lambda$ on X for every $\lambda \in \mathfrak{h}^*$. The $\mathscr{D}_\lambda$-modules on X have cohomology groups which are Harish-Chandra modules with infinitesimal character $[\lambda] \in \mathfrak{h}^*/\mathcal{W}$. When τ and λ are compatible, we define the *standard module* on X corresponding to the pair (τ, λ) to be the $\mathscr{D}_\lambda$-module direct image $i_+\tau$ of τ along the inclusion $i : Q \to X$. Recall for V a $(\mathfrak{b}, L)$-module with $\mathfrak{b} \subset \mathfrak{g}$ a Borel subalgebra and L a subgroup of K, we induce V to a $(\mathfrak{g}, L)$-module by taking the tensor product $\mathrm{ind}_{\mathfrak{b},L}^{\mathfrak{g},L}(V) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} V$. Let T_x denote the geometric fiber functor.

We state the main theorem of [5] not in its original form as a duality statement, but instead without contragredients so that it takes a form similar to the natural formulation of our main result.

Theorem 1.1 ([5], Theorem 4.3). *Let $x \in Q$ be any point, let B_x be its stabilizer in G, and let $\mathfrak{b}_x$ be the Lie algebra of B_x. Put $\mathfrak{n}_x = [\mathfrak{b}_x, \mathfrak{b}_x]$ and let $\bar{\mathfrak{n}}_x$ be its opposite in $\mathfrak{g}$. Then for all $p \in \mathbb{Z}$, we have*

$$\mathrm{H}^p(X, i_+\tau) \simeq \mathbf{R}^{d_Q+p}\Gamma_{K,B_x\cap K}(\mathrm{ind}_{\mathfrak{b}_x,B_x\cap K}^{\mathfrak{g},B_x\cap K}(T_x\tau \otimes \wedge^{\mathrm{top}}\bar{\mathfrak{n}}_x))$$

as $\mathcal{U}(\mathfrak{g})$-modules with infinitesimal character $[\lambda]$.

This theorem shows that the sheaf cohomology of standard K-equivariant $\mathscr{D}_\lambda$-modules on the full flag variety X for are isomorphic to cohomologically induced modules; that is, modules which are cohomologically induced from Borels. Our main result is the analogous identification of the cohomology of standard $\mathscr{D}_\lambda$-modules on a partial flag variety X_θ, where θ is a subset of simple roots, with Harish-Chandra modules cohomologically induced from parabolics of type θ.

Unfortunately, Theorem 1.1. fails to generalize immediately to partial flag varieties because the direct image functor i_+ is not necessarily exact for the inclusion of a non-affinely embedded K-orbit Q in X_θ. That is, the direct image $i_+\tau$ may be a complex of $\mathscr{D}_\lambda$ modules rather than a single sheaf. Theorem 1.2. below is an extension of Theorem 1.1. which incorporates the possible failure of exactness of i_+. Theorem 1.1. can be recovered as an immediate corollary. Let X_θ be a partial flag variety for $\mathfrak{g}$ and $p : X \to X_\theta$ the natural projection from the full flag variety. Let ρ be the half-sum of roots in Σ^+, let ρ_θ be the half-sum of roots in Σ^+ generated by θ, and define $\rho_n = \rho - \rho_\theta$. Our main theorem is then:

Theorem 1.2 (Main Theorem). *Let $\mathscr{D}_\lambda$ be a homogeneous tdo on X_θ and let τ be a connection on a K-orbit Q compatible with $\lambda + \rho_n$. For $x \in Q$, let $\mathfrak{p}_x$ be the corresponding parabolic in $\mathfrak{g}$, let $\mathfrak{n}_x = [\mathfrak{p}_x, \mathfrak{p}_x]$, and let S_x be the stabilizer of x in K. Then there is an isomorphism*

$$\mathbf{R}\Gamma(X, p^\circ i_+\tau) \simeq \Gamma^{\mathrm{equi}}_{K,S_x}(\mathrm{ind}^{\mathfrak{g},S_x}_{\mathfrak{p}_x,S_x}(T_x\tau \otimes \wedge^{\mathrm{top}}\bar{\mathfrak{n}}_x))[d_Q] \tag{1}$$

in $\mathrm{D^b}(\mathcal{U}_{[\lambda-\rho_\theta]}, K)$, where d_Q is the dimension of Q. Upon taking cohomology, there is a convergent spectral sequence

$$\mathbf{R}^p\Gamma(X, p^\circ \mathbf{R}^q i_+\tau) \Longrightarrow \mathbf{R}^{d_Q+p+q}\Gamma_{K,S_x}(\mathrm{ind}^{\mathfrak{g},S_x}_{\mathfrak{p}_x,S_x}(T_x\tau \otimes \wedge^{\mathrm{top}}\bar{\mathfrak{n}}_x)). \tag{2}$$

In this theorem, the category $\mathrm{D^b}(\mathcal{U}_\chi, K)$ is the equivariant bounded derived category of Harish-Chandra modules with infinitesimal character χ and $\Gamma^{\mathrm{equi}}_{K,S}$ is the equivariant Zuckerman functor introduced in §3.

The spectral sequence (2) collapses in special cases, such as when X_θ is the full flag variety, but in general Theorem 1.2. is the closest we get to a direct generalization of Theorem 1.1. However, for applications to composition series computations in the Grothendieck group, the convergence of (2) is sufficient.

The idea behind the proof of Theorem 1.2. is that the standard sheaf $i_+\tau$ is determined entirely by the geometric fiber $T_x\tau$ at a point $x \in Q$. We make this precise by constructing an essential inverse to the functor T_x. In this construction we introduce the geometric Zuckerman functor $\Gamma^{\mathrm{geo}}_{K,S}$. The isomorphism (1) follows from the commutivity properties of $\Gamma^{\mathrm{geo}}_{K,S}$, together with Theorem 1.3. below, which allows us to identify the $\mathcal{U}(\mathfrak{g})$-module structure on the sheaf cohomology in Theorem 1.2.

Theorem 1.3 (Embedding Theorem). *The inverse image functor $p^\circ : \mathcal{M}(\mathscr{D}_\lambda) \to \mathcal{M}(\mathscr{D}^p_\lambda)$ is fully faithful for all λ, and for λ anti-dominant, we have $\Gamma \circ p^\circ = p^* \circ \Gamma$, where $p^* : \mathcal{M}(\Gamma(\mathscr{D}_\lambda)) \to \mathcal{M}(\Gamma(\mathscr{D}^p_\lambda))$ is the usual pull-back of modules induced by the natural map $\Gamma(\mathscr{D}^p_\lambda) \to \Gamma(\mathscr{D}_\lambda)$.*

1.2. **Contents of Paper.** In §2-3, we review twisted differential operators on homogeneous spaces and the construction of the equivariant Zuckerman functor $\Gamma^{\mathrm{equi}}_{K,S}$ of [13]. The functor $\Gamma^{\mathrm{equi}}_{K,S}$ is the generalization of the usual derived Zuckerman functor to categories of derived equivariant complexes. In [13], Pandžić proves that by taking cohomology of $\Gamma^{\mathrm{equi}}_{K,S}$ we recover the usual Zuckerman functors. That is, for all p we have

$$\mathrm{H}^p(\Gamma^{\mathrm{equi}}_{K,T} V^\bullet) = \mathbf{R}^p\Gamma_{K,T}(V^\bullet). \tag{3}$$

Section 4 is the technical heart of the paper where we introduce the derived equivariant category of Harish-Chandra sheaves, define the *geometric Zuckerman functor* $\Gamma^{\mathrm{geo}}_{K,S}$ (which is the localization of $\Gamma^{\mathrm{equi}}_{K,S}$), and prove that $\Gamma^{\mathrm{geo}}_{K,S}$ has sundry properties that will be used in the proof of the Theorem 1.2. In the final section, we prove Theorems 1.2. and 1.3. and end the paper with a brief reformulation of Theorem 1.2. as a duality statement.

1.3. **Acknowledgements.** I would first like to thank my advisor Dragan Miličić, without whom none of this would have been possible, and Pavle Pandžić, whose thesis provided the groundwork for mine. Special thanks go to Wolfgang Soergel and Andrew Snowden for all of their useful comments. I would also like to thank the members of the University of Utah math department for their support during my graduate student years and express my gratitude to the NSF, as I was fortunate enough to have been supported by the VIGRE grant for several semesters.

2. Twisted Sheaves of Differential Operators

In this section, we introduce our notation for the direct and inverse image of $\mathscr{D}$-modules, where $\mathscr{D}$ is a twisted sheaf of differential operators. Additionally, we give classification results for homogeneous sheaves of twisted differential operators on generalized flag varieties. We learned much of the material from Miličić's unpublished notes [9] and [10].

2.1. **Definitions.** We will always use $\mathscr{D}_X$ to denote the sheaf of differential operators on a smooth complex algebraic variety X and more generally $\mathscr{D}$ for a twisted sheaf of differential operators (tdo); that is, a sheaf of $\mathscr{O}_X$-algebras locally isomorphic to $\mathscr{D}_X$. Let $\mathcal{M}(\mathscr{D})$ denote the category of left $\mathscr{D}$-modules and $\mathrm{D^b}(\mathscr{D})$ the corresponding bounded derived category. For right $\mathscr{D}$-modules we write $\mathcal{M}(\mathscr{D})_{\mathrm{r}}$ and $\mathrm{D^b}(\mathscr{D})_{\mathrm{r}}$, respectively.

Fix a smooth map $f : Y \to X$ between smooth varieties and define $\mathscr{D}^f$ to be the sheaf of differential endomorphisms of the left $\mathscr{O}_Y$-, right $f^{-1}\mathscr{D}$-module $\mathscr{D}_{Y\to X} = f^*\mathscr{D}$. This sheaf of operators $\mathscr{D}^f$ is itself a tdo on Y. In the trivial example, we have $\mathscr{D} = \mathscr{D}_X$ and $\mathscr{D}^f = \mathscr{D}_Y$ for any f. For maps $f : X \to Y$ and $g : Y \to Z$ and a tdo $\mathscr{D}$ on Z, we have $(\mathscr{D}^g)^f \simeq \mathscr{D}^{g\circ f}$.

2.2. **Inverse Image.** Let $f : Y \to X$ and $\mathscr{D}$ be as in the above section. We denote the inverse image functor from $\mathcal{M}(\mathscr{D})$ to $\mathcal{M}(\mathscr{D}^f)$ by f°. It is defined as

$$f^\circ(\text{ - }) := \mathscr{D}_{Y\to X} \otimes_{f^{-1}\mathscr{D}} f^{-1}(\text{ - }).$$

Here f^{-1} is the usual sheaf inverse image. The functor f° is right exact, exact when f is flat, and has finite left cohomological dimension.

The category $\mathcal{M}(\mathscr{D})$ has enough projectives, and so the derived inverse image functor

$$\mathbf{L}f^\circ : \mathrm{D^b}(\mathscr{D}) \to \mathrm{D^b}(\mathscr{D}^f)$$

exists. In [4], Borel defines the functor

$$f^! := \mathbf{L}f^\circ[d_{Y/X}] : \mathrm{D^b}(\mathscr{D}) \to \mathrm{D^b}(\mathscr{D}^f),$$

where $d_{Y/X} = \dim Y - \dim X$. Introducing the shift by $d_{Y/X}$ guarantees the functor $f^!$ behaves well with respect to Verdier duality.

2.3. **Direct Image.** Again let $f : Y \to X$ be as in §2.1 and let $\mathscr{D}$ be a tdo on X. We will define the direct image functor f_+, then examine this functor for f a surjective submersion. The opposite sheaf $\mathscr{D}^\circ$ of any tdo $\mathscr{D}$ is again a tdo [10, Prop. 11]. There is an *isomorphism* of categories $\mathcal{M}(\mathscr{D}^\circ)_{\mathrm{r}} \simeq \mathcal{M}(\mathscr{D})$, which is the identity on objects. Let $\omega_{Y/X}$ denote the relative canonical bundle for f.

Definition 2.1. *Up to conjugation by the isomorphism $\mathcal{M}(\mathscr{D}) \simeq \mathcal{M}(\mathscr{D}^\circ)_{\mathrm{r}}$, the direct image functor $f_+ : \mathrm{D}^b(\mathscr{D}^f) \to \mathrm{D}^b(\mathscr{D})$ is defined by*

$$f_+(\text{ - }) = \mathbf{R}f_*(\text{ - } \otimes \omega_{Y/X} \otimes^{\mathbf{L}}_{(\mathscr{D}^\circ)^f} \mathscr{D}_{Y\to X}).$$

This definition is the translation to left $\mathscr{D}$-modules of the usual construction:

$$f_+ : \mathrm{D^b}(\mathscr{D}^f)_{\mathrm{r}} \to \mathrm{D^b}(\mathscr{D})_{\mathrm{r}}, \quad f_+(\text{ - }) = \mathbf{R}f_*(\text{ - } \otimes^{\mathbf{L}}_{\mathscr{D}^f} \mathscr{D}_{Y\to X}).$$

In general, the direct image f_+ is neither right nor left exact. However, if f is an affine morphism, then f_* is exact and thus f_+ is right exact. If $\mathscr{D}_{Y\to X}$ is a flat $\mathscr{D}^f$-module, such as when f is an immersion, then the tensor product is exact, so f_+ is left exact. Putting these two special cases together we find that if f an affine immersion, then f_+ is exact. Moreover, if f is a closed immersion, then $f^!$ is the right adjoint to f_+.

Let f be a surjective submersion. In this case, there is a locally free left $\mathscr{D}^f$-, right $f^{-1}\mathscr{O}_X$-module resolution $\mathcal{T}^\bullet_{Y/X}(\mathscr{D}^f)$ of $\mathscr{D}_{Y\to X}$ given by

$$\mathcal{T}^{-k}_{Y/X}(\mathscr{D}^f) = \mathscr{D}^f \otimes_{\mathscr{O}_Y} \wedge^k \mathscr{T}_{Y/X}, \quad k \in \mathbb{Z},$$

with the usual de Rham differential. Here $\mathscr{T}_{Y/X} := \Omega^*_{Y/X}$ is the sheaf of local vector fields tangent to the fibers of f. Note $\mathscr{T}_{Y/X} \subset \mathscr{D}^f$ since the twist of $\mathscr{D}^f$ is trivial along these fibers. The direct image with respect to this resolution gives

$$\begin{aligned} f_+(\mathscr{V}) &= \mathbf{R}f_*(\mathscr{V} \otimes \omega_{Y/X} \otimes_{(\mathscr{D}^\circ)^f} \mathcal{T}^\bullet_{Y/X}(\mathscr{D}^\circ)^f) \\ &= \mathbf{R}f_*(\Omega^\bullet_{Y/X}(\mathscr{D}^f) \otimes_{\mathscr{D}^f} \mathscr{V})[d_{Y/X}] \end{aligned}$$

for all $\mathscr{V} \in \mathcal{M}(\mathscr{D}^f)$, where $\Omega^\bullet_{Y/X}(\mathscr{D}^f)$ is the relative de Rham complex tensored with $\mathscr{D}^f$. In this case, it is transparent that $f_+[-d_{Y/X}]$ is the right adjoint of f°.

2.4. **Homogeneous Twisted Sheaves of Differential Operators.** In this section we classify homogeneous sheaves of twisted differential operators on generalized flag varieties. The content follows the analogous constructions in [10]. The generalized flag varieties are homogeneous spaces X for a complex reductive group G. We consider only tdo's which are equivariant with respect to the G-action on X; more precisely, we will work exclusively with *homogeneous* twisted sheaves of differential operators.

Definition 2.2. *A* homogeneous tdo *on a complex G-variety X is a tdo $\mathscr{D}$ with a G-equivariant structure γ and a morphism of algebras $\alpha : \mathcal{U}(\mathfrak{g}) \to \Gamma(X, \mathscr{D})$ satisfying:*
(H1). *The group G acts on $\mathscr{D}$ by algebra homomorphisms.*
(H2). *The differential of γ agrees with the adjoint action — that is,*

$$d\gamma_\xi(T) = [\alpha(\xi), T], \quad \forall\, \xi \in \mathfrak{g},\ T \in \mathscr{D}.$$

(H3). *The map α is G-equivariant.*

We now classify homogeneous tdo's on a generalized flag variety X of a complex reductive Lie group G. Let $\mathfrak{h}$ be the abstract Cartan for $\mathfrak{g}$ and let θ be the subset of simple positive roots corresponding to X. If $x \in X$ is any point and $\mathfrak{p}_x$ the parabolic determined by x, define

$$\mathfrak{h}_\theta = \mathfrak{p}_x/[\mathfrak{p}_x, \mathfrak{p}_x].$$

Proposition 2.3. *The space $\mathfrak{h}^*_\theta$ parameterizes isomorphism classes of homogeneous tdo's on the partial flag variety X of type θ.*

This proposition is a special case of [10, Theorem 1.2.4]. The proof is constructive; for completeness, we outline the construction of the homogeneous tdo $\mathscr{D}_{X,\lambda}$ for any $\lambda \in \mathfrak{h}^*_\theta$. Let $\mathfrak{g}^\circ$ denote the trivial bundle $\mathscr{O}_X \otimes_{\mathbb{C}} \mathfrak{g}$. There is a surjection $\mathfrak{g}^\circ \to \mathscr{T}_X$ with kernel $\mathfrak{p}^\circ$, which has geometric fiber $T_x\mathfrak{p}^\circ = \mathfrak{p}_x$ at $x \in X$, where $\mathfrak{p}_x$ is the parabolic corresponding to x. Let P_x be the stabilizer of x in G so that P_x has $\mathfrak{p}_x$ as its Lie algebra. Any $\lambda \in \mathfrak{p}^*_x$ which is P_x-invariant determines a G-equivariant morphism $\lambda^\circ : \mathfrak{p}^\circ \to \mathscr{O}_X$. In fact, these morphisms are in bijection with P_x-invariant linear forms on $\mathfrak{p}_x$. Define $\mathcal{U}^\circ := \mathscr{O}_X \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{g})$ and the map $\phi_\lambda : \mathfrak{p}^\circ \to \mathcal{U}^\circ$ by $\phi_\lambda(s) = s - \lambda^\circ(s)$ for $s \in \mathfrak{p}^\circ$. The image of ϕ_λ generates a two-sided ideal $\mathscr{I}_\lambda$ in $\mathcal{U}^\circ$; finally, define

$$\mathscr{D}_{X,\lambda} := \mathcal{U}^\circ/\mathscr{I}_\lambda.$$

The action of G on $\mathcal{U}(\mathfrak{g})$ induces an algebraic action on $\mathscr{D}_{X,\lambda}$, and similarly, the surjection $\mathcal{U}^\circ \to \mathscr{D}_{X,\lambda}$ determines a morphism

$$\alpha : \mathcal{U}(\mathfrak{g}) \to \Gamma(X, \mathscr{D}_{X,\lambda})$$

upon taking global sections. That the G-action and α satisfy $(H1)$–$(H3)$ is obvious, and therefore, $\mathscr{D}_{X,\lambda}$ is a homogeneous twisted sheaf of differential operators.

2.5. **The Infinitesimal Character of $\mathscr{D}_{X,\lambda}$.** In this section we compute the infinitesimal character of $\Gamma(X, \mathscr{D}_{X,\lambda})$. Let $[\lambda] \in \mathfrak{h}^*/\mathcal{W}$ be the $\mathcal{W}$-orbit of $\lambda \in \mathfrak{h}^*$. Recall that when X is the full flag variety, for any $\lambda \in \mathfrak{h}^*$ there is an isomorphism $\Gamma(X, \mathscr{D}_{X,\lambda}) \simeq \mathcal{U}_{[\lambda-\rho]}$ and all higher cohomology vanishes. Consequently, we define $\mathscr{D}_\mu := \mathscr{D}_{X,\mu+\rho}$ to compensate for the ρ-shift in the infinitesimal character of global sections.

Unfortunately, the global sections of $\mathscr{D}_{X,\lambda}$ for X a partial flag variety do not always appear as a quotient of $\mathcal{U}(\mathfrak{g})$. However, we can determine the infinitesimal character without computing global sections explicitly. Define

$$\mathscr{D}_{\mathfrak{h}_\theta} = \mathcal{U}^\circ/[\mathfrak{p}^\circ, \mathfrak{p}^\circ]\mathcal{U}^\circ.$$

The quotient $\mathfrak{h}_\theta^\circ = \mathfrak{p}^\circ/D\mathfrak{p}^\circ$ is the trivial bundle $\mathfrak{h}_\theta^\circ = \mathscr{O}_X \otimes_{\mathbb{C}} \mathfrak{h}_\theta$. For $\lambda \in \mathfrak{h}_\theta^*$, the corresponding morphism $\phi_\lambda : \mathfrak{p}^\circ \to \mathcal{U}^\circ$ defining $\mathscr{D}_{X,\lambda}$ descends to $\phi_\lambda : \mathfrak{h}_\theta^\circ \to \mathscr{D}_{\mathfrak{h}_\theta}$. The quotient $\mathfrak{h} \to \mathfrak{h}_\theta$ allows us to extend ϕ_λ to $\mathcal{U}(\mathfrak{h})$, and then compose with the abstract Harish-Chandra isomorphism to get a map $\mathcal{Z}(\mathfrak{g}) \to \mathscr{D}_{\mathfrak{h}_\theta}$. Let ρ_θ and ρ_n be defined as in the introduction. Note ρ_θ vanishes in the projection of $\mathfrak{h}^*$ to the subspace $\mathfrak{h}_\theta^*$. Define

$$\mathscr{D}_\lambda = \mathscr{D}_{X,\lambda+\rho_n}.$$

Then, the global sections of the tdo $\mathscr{D}_\lambda$ has infinitesimal character $[\lambda - \rho_\theta] \in \mathfrak{h}^*/\mathcal{W}$.

We end with results illustrating some relationships between the twisting parameters for various homogeneous tdo's. Let $p : X \to X_\theta$ be the projection of the full flag variety X to the partial flag variety X_θ of type θ. There is then an equality $\mathscr{D}^p_{X,\lambda} = \mathscr{D}_{X,\lambda}$ and so

$$\mathscr{D}^p_\lambda = \mathscr{D}_{\lambda-\rho_\theta}.$$

Also, since the opposite tdo appears in the construction of the direct image, we include the following proposition.

Proposition 2.4. *Let $\mathscr{D}_\lambda$ be any homogeneous tdo on the partial flag variety X_θ. Then,*

$$\mathscr{D}^\circ_\lambda = \mathscr{D}_{-\lambda}.$$

Equivalently, we have $\mathscr{D}^\circ_{X_\theta,\lambda} = \mathscr{D}_{X_\theta,-\lambda+2\rho_n}$.

2.6. **Anti-dominance and $\mathscr{D}$-affineness.** In this section we give some vanishing results for cohomology of $\mathscr{D}$-modules on generalized flag varieties. Let $\mathfrak{g}$ be a complex semi-simple Lie algebra, with abstract Cartan triple $(\mathfrak{h}, \Sigma, \Sigma^+)$. We will use $\Sigma^\vee$ to denote the co-roots in $\mathfrak{h}$. For $\lambda \in \mathfrak{h}^*$, we say λ is *anti-dominant* if $\alpha^\vee(\lambda)$ is not a positive integer for all $\alpha \in \Sigma^+$. Further, we say λ is *regular* if the $\alpha^\vee(\lambda)$ are all non-zero as well. If $\theta \subset \Pi^+$ is a subset of simple roots, let Σ^+_θ denote the closure of θ in Σ^+ under addition. Define $\Sigma_n := \Sigma^+ \backslash \Sigma^+_\theta$. For $\mathfrak{p}$ a parabolic of type θ, any specialization of $(\mathfrak{h}, \Sigma, \Sigma^+)$ to a Cartan triple for $\mathfrak{p}$ will send Σ^+_θ to positive roots contained in a Levi factor of $\mathfrak{p}$ and Σ_n to the roots of the nilradical of $\mathfrak{p}$. Let ρ_θ and ρ_n be the half-sum of positive roots in Σ^+_θ, respectively Σ_n. Since $\mathfrak{h}^*_\theta$ naturally embeds to a subspace of $\mathfrak{h}^*$, we can define anti-dominance on $\mathfrak{h}^*_\theta$ by restricting the condition on $\mathfrak{h}^*$. However, it will be more useful to include a shift in the definition.

Definition 2.5. *The character $\lambda \in \mathfrak{h}^*_\theta$ is* anti-dominant *if $\lambda - \rho_\theta \in \mathfrak{h}^*$ is. Likewise, $\lambda \in \mathfrak{h}^*_\theta$ is* regular *if $\lambda - \rho_\theta \in \mathfrak{h}^*$ is.*

If θ is empty, $\mathfrak{h}_\theta = \mathfrak{h}$ and $\rho_\theta = 0$, so this generalized definition is consistent with the original. From [1], we have the following definition and results.

Definition 2.6. *Let X be a generalized flag variety and $\mathscr{D}$ a tdo on X. Say X is* $\mathscr{D}$-affine *if for every $\mathscr{F} \in \mathcal{M}(\mathscr{D})$ we have $\Gamma(X, \mathscr{F})$ generated by global sections and $\mathrm{H}^i(X, \mathscr{F}) = 0$ for all $i > 0$.*

Proposition 2.7. *If X is $\mathscr{D}$-affine, the global sections functor*

$$\Gamma : \mathcal{M}(\mathscr{D}) \to \mathcal{M}(\mathcal{D})$$

is an equivalence of categories, where $\mathcal{D} = \Gamma(X, \mathscr{D})$.

Anti-dominance of λ is necessary for $\mathscr{D}_\lambda$-affineness of the full flag variety X; see for example [11] or [1]. Note our convention of positive roots is the opposite of [1]; i.e., for them dominance rather than anti-dominance of λ determines $\mathscr{D}_\lambda$-affineness.

Theorem 2.8. *Let X be the full flag variety, $\lambda \in \mathfrak{h}^*$.*

(1) *If λ is dominant, then $\Gamma : \mathcal{M}(\mathscr{D}_\lambda) \to \mathcal{M}(\mathcal{U}_{[\lambda]})$ is exact.*
(2) *If λ is also regular, then Γ is faithful.*

A consequence of this theorem is that for λ anti-dominant and regular, Γ gives an equivalence of categories. Its quasi-inverse Δ_λ sends a $\mathcal{U}_{[\lambda]}$-module V to

$$\Delta_\lambda(V) = \mathscr{D}_\lambda \otimes_{\mathcal{U}_{[\lambda]}} V.$$

We prove the following proposition in §5.1.

Proposition 2.9. *Let $\lambda \in \mathfrak{h}_\theta^*$ be anti-dominant and regular. Then X_θ is $\mathscr{D}_\lambda$-affine.*

3. The Equivariant Zuckerman Functor

In this section, we recall the main definitions and some results of the thesis of Pandžić [13], including the construction of the equivariant Zuckerman functor.

3.1. $(\mathcal{A}, K)$**-Modules.** Let $(\mathcal{A}, K)$ be a pair consisting of an associative algebra $\mathcal{A}$ over $\mathbb{C}$ and K a complex algebraic group. The algebra $\mathcal{A}$ is equipped with an algebraic K-action ϕ, and a K-equivariant Lie algebra morphism $\psi : \mathfrak{k} \to \mathcal{A}$ such that

$$d\phi(\xi)(a) = [\psi(\xi), a], \quad \xi \in \mathfrak{k},\ a \in \mathcal{A}.$$

Such pairs are called *Harish-Chandra pairs*. We will eventually take $\mathcal{A}$ to be global sections of a tdo on a generalized flag variety.

Definition 3.1. *A* weak $(\mathcal{A}, K)$-module *is a triple (V, π, ν) consisting of*
(1) *V an $\mathcal{A}$-module with action π, and*
(2) *V an algebraic K-module with action ν, such that*
(3) *the $\mathcal{A}$-action map $\mathcal{A} \otimes V \to V$ is K-equivariant. In other words,*

$$\nu(k)\pi(a)\nu(k^{-1}) = \pi(\phi(k)a)$$

for all $k \in K$ and $a \in \mathcal{A}$.

An $(\mathcal{A}, K)$-module *is a weak $(\mathcal{A}, K)$-module such that*
(4) *$d\nu = \pi \circ \psi$.*

This definition generalizes the notion of (weak) Harish-Chandra modules for the pair $(\mathfrak{g}, K)$.

Let $\mathcal{M}_w(\mathcal{A}, K)$ be the category of all weak $(\mathcal{A}, K)$-modules. Morphisms of weak $(\mathcal{A}, K)$-modules are linear maps compatible with both the $\mathcal{A}$- and K-module structures. Similarly, denote by $\mathcal{M}(\mathcal{A}, K)$ the category of $(\mathcal{A}, K)$-modules. Let $\mathcal{C}(\mathcal{M}_{(w)}(\mathcal{A}, K))$ and $\mathcal{K}(\mathcal{M}_{(w)}(\mathcal{A}, K))$ denote the category of complexes and homotopy category of complexes of (weak) $(\mathcal{A}, K)$-modules, respectively. The derived category $\mathrm{D}(\mathcal{M}_{(w)}(\mathcal{A}, K))$ of (weak) $(\mathcal{A}, K)$-modules is constructed in the usual way, by localizing $\mathcal{K}(\mathcal{M}_{(w)}(\mathcal{A}, K))$ with respect to quasi-isomorphisms. Therefore for weak modules, we may simplify our notation by using $\mathcal{C}_w(\mathcal{A}, K)$, etc.

3.2. **Equivariant Derived Categories.** Rather than working in the triangulated categories derived directly from the abelian categories $\mathcal{M}(\mathcal{U}_\chi, K)$ (for some $\chi \in \mathfrak{h}^*/\mathcal{W}$), for the purposes of localization it is necessary to work with the equivariant derived category. We give the needed definitions here.

Definition 3.2. *An* equivariant $(\mathcal{A}, K)$-complex *is a pair $(V^\bullet, i)$ with $V^\bullet$ a complex of weak $(\mathcal{A}, K)$-modules, and i is a linear map from $\mathfrak{k}$ to graded linear degree -1 endomorphisms of $V^\bullet$ satisfying:*
(1) *The i_ξ are $\mathcal{A}$-morphisms for all $\xi \in \mathfrak{k}$.*
(2) *The i_ξ are K-equivariant for all $k \in K$.*
(3) *The sum $i_\xi i_\eta + i_\eta i_\xi = 0$ for all $\eta, \xi \in \mathfrak{k}$.*
(4) *For every $\xi \in \mathfrak{k}$, the sum $di_\xi + i_\xi d = \omega(\xi)$ where $\omega = \nu - \pi$.*

We summarize conditions (1) and (2) by stating $i \in \mathrm{Hom}_K(\mathfrak{k}, \mathrm{Hom}_{\mathcal{A}}(V^\bullet, V^\bullet[-1]))$, where we take $\mathrm{Hom}_{\mathcal{A}}(V^\bullet, V^\bullet[-1])$ in the category of graded $\mathcal{A}$-modules, and use the conjugation action of K. Specifically, we have K acting on $f \in \mathrm{Hom}_{\mathcal{A}}(V^\bullet, V^\bullet[-1])$ by

$$(k.f)(v) = \nu(k).f(\nu(k^{-1})v)$$

for all $k \in K$ and $v \in V^\bullet$. The fourth condition implies the cohomology modules of $V^\bullet$ are $(\mathcal{A}, K)$-modules.

A morphism of equivariant $(\mathcal{A},K)$-complexes is a morphism of complexes of weak $(\mathcal{A},K)$-modules which commutes with i_ξ for all $\xi \in \mathfrak{k}$. The category $\mathcal{C}(\mathcal{A},K)$ of equivariant $(\mathcal{A},K)$-complexes is abelian. Two morphisms

$$\phi,\psi : (V^\bullet, i) \to (W^\bullet, i)$$

are homotopic if there exists a homotopy of complexes $h : V^\bullet \to W^\bullet[-1]$ which anti-commutes with i_ξ for all $\xi \in \mathfrak{k}$. That is,

$$h \circ i_\xi = -i_\xi \circ h.$$

Let $\mathcal{K}(\mathcal{A},K)$ be the homotopy category of equivariant $(\mathcal{A},K)$-complexes and $\mathrm{D}(\mathcal{A},K)$ its localization by quasi-isomorphisms. The category $\mathrm{D}(\mathcal{A},K)$ is known as the equivariant derived category of $(\mathcal{A},K)$-modules.

For modules $V^\bullet$ and $W^\bullet \in \mathcal{C}_w(\mathcal{A},K)$, define the homomorphism complex (*Hom-complex*) by setting

$$\mathrm{Hom}^k(V^\bullet, W^\bullet) = \prod_p \mathrm{Hom}_{\mathcal{M}_w(\mathcal{A},K)}(V^p, W^{p+k})$$

with differential $d^k(f) = d_W \circ f - (-1)^k f \circ d_V$. Clearly then $\mathrm{Hom}^0(V^\bullet, W^\bullet) = \mathrm{Hom}_{\mathcal{C}_w(\mathcal{A},K)}(V^\bullet, W^\bullet)$ and $\mathrm{H}^0(\mathrm{Hom}^\bullet(V^\bullet, W^\bullet)) = \mathrm{Hom}_{\mathcal{K}_w(\mathcal{A},K)}(V^\bullet, W^\bullet)$. The Hom-complex for objects $(V^\bullet, i)$ and $(W^\bullet, j)$ in $\mathcal{C}(\mathcal{A},K)$ is defined in the same way, but with morphisms $f \in \mathrm{Hom}^k(V^\bullet, W^\bullet)$ such that

$$f i_\xi = (-1)^k j_\xi f, \quad \forall\, \xi \in \mathfrak{k}.$$

Again, we have $\mathrm{Hom}^0(V^\bullet, W^\bullet) = \mathrm{Hom}_{\mathcal{C}(\mathcal{A},K)}(V^\bullet, W^\bullet)$ and $\mathrm{H}^0(\mathrm{Hom}^\bullet(V^\bullet, W^\bullet)) = \mathrm{Hom}_{\mathcal{K}(\mathcal{A},K)}(V^\bullet, W^\bullet)$.

Let $(*) = \mathcal{C}, \mathcal{K}$ or D. There is a functor $\mathrm{For_h}$ from $(*)(\mathcal{A},K)$ to $(*)_w(\mathcal{A},K)$ which forgets the homotopy i for the object $(V^\bullet, i)$. Obviously $\mathrm{For_h} : \mathcal{C}(\mathcal{A},K) \to \mathcal{C}_w(\mathcal{A},K)$ is faithful, but the same cannot necessarily be said for the homotopy or derived categories.

For the pair $(\mathfrak{g}, K)$ and $\mathcal{A} = \mathcal{U}(\mathfrak{g})$ (or more generally a quotient of $\mathcal{U}(\mathfrak{g})$), an important example of an equivariant complex in $\mathcal{C}^{\mathrm{b}}(\mathcal{A},K)$ is the *standard complex* $\mathcal{N}(\mathfrak{g})$ of $\mathfrak{g}$. The complex underlying $\mathcal{N}(\mathfrak{g})$ is the Koszul resolution of $\mathbb{C}$ as a $\mathcal{U}(\mathfrak{g})$-module. That is, for any integer k, we have

$$\mathcal{N}(\mathfrak{g})^{-k} = \mathcal{U}(\mathfrak{g}) \otimes_{\mathbb{C}} \wedge^k \mathfrak{g}.$$

If $u \otimes \tau \in \mathcal{N}(\mathfrak{g})^{-(k+1)}$ with $\tau = \tau_0 \wedge \ldots \wedge \tau_k$, the Koszul differential in degree $-(k+1)$ is

$$\begin{array}{rcl} d^{-(k+1)}(u \otimes \tau) & = & \sum_{i=0}^k (-1)^i u\tau_i \otimes \tau_0 \wedge \ldots \hat{\tau}_i \ldots \wedge \tau_k \\ & + & \sum_{0 \le i < j \le k} (-1)^{i+j} u \otimes [\tau_i, \tau_j] \wedge \tau_0 \wedge \ldots \hat{\tau}_i \ldots \hat{\tau}_j \ldots \wedge \tau_k. \end{array}$$

The action π_N of $\mathfrak{g}$ in any degree is by left multiplication on $\mathcal{U}(\mathfrak{g})$. The action ν_N of K is induced on each side of the tensor product by the map $\phi : K \to \mathrm{Int}(\mathfrak{g})$, and its differential

$$d\nu_N(\xi)(u \otimes \tau) = d\phi(\xi)u \otimes \tau + u \otimes d\phi(\xi)\tau.$$

There is a natural homotopy i of these actions on $\mathcal{N}(\mathfrak{g})$ given in the following proposition.

Proposition 3.3. *For any $\xi \in \mathfrak{k}$, $u \otimes \tau \in \mathcal{N}(\mathfrak{g})^{-k}$, define $i_\xi(u \otimes \tau) = -u \otimes \psi(\xi) \wedge \tau$. Then, we have $(\mathcal{N}(\mathfrak{g}), i) \in \mathcal{C}^{\mathrm{b}}(\mathcal{A},K)$.*

The proof is a straightforward check, which we omit.

3.3. **The Right Adjoint.** The results of [13] can be stated in terms of the right adjoint $\mathrm{Ind_h}$ to $\mathrm{For_h}$ defined below. For the geometric constructions of sections 4 and 5, an alternative definition using tensor products will be more useful. We give both definitions and show they are equivalent.

Define $\mathrm{Ind_h} : \mathcal{C}_w(\mathcal{A},K) \to \mathcal{C}(\mathcal{A},K)$ to take $V^\bullet \in \mathcal{C}_w(\mathcal{A},K)$ to $\mathrm{Ind_h}(V^\bullet) = \mathrm{Hom}^\bullet(\mathcal{N}(\mathfrak{k}), V^\bullet)$ with $f \circ \pi_N(\xi) = \omega_V(\phi(\xi)) \circ f$ for all $f \in \mathrm{Ind_h}(V^\bullet)$. The pair $(\mathcal{A},K)$ acts on a map $f : \mathcal{N}(\mathfrak{k}) \to V^\bullet$ by

$$\begin{array}{rcll} \pi(X)f & = & \pi_V(X) \circ f, & \forall\, X \in \mathcal{A} \\ \nu(k)f & = & \nu_V(k) \circ f \circ \nu_N(k^{-1}), & \forall\, k \in K \end{array}$$

where (π_V, ν_V) denote the $(\mathcal{A},K)$-actions on $V^\bullet$. Then, for all $\xi \in \mathfrak{k}$, $\omega(\xi)f = -f \circ \omega_N(\xi)$. There is a natural homotopy i on $\mathrm{Ind_h}(V^\bullet)$ given by $i_\xi f = (-1)^{k-1} f \circ i_\xi$ for f a degree k homomorphism, and any $\xi \in \mathfrak{k}$. The following proposition can be proved by direct computation.

Proposition 3.4. *For any $V^\bullet \in \mathcal{C}_w(\mathcal{A},K)$, the pair $(\mathrm{Ind_h}(V^\bullet), i) \in \mathcal{C}(\mathcal{A},K)$.*

That $\mathrm{Ind_h}$ is right adjoint to the forgetful functor $\mathrm{For_h}$ is proved in detail in [7, Prop. 4.2.3].

We can construct the right adjoint alternatively using a tensor product. For $V^\bullet \in \mathcal{C}_w(\mathcal{A}, K)$, let ω_V denote the $\mathfrak{k}$-action on $V^\bullet$ extended to $\mathcal{U}(\mathfrak{k})$. Then, $V^\bullet$ can be made into a right $\mathcal{U}(\mathfrak{k})$-module by letting $u \in \mathcal{U}(\mathfrak{k})$ act on $v \in V^\bullet$ by $-\omega_V(u^\iota)v$, where ι is the principal anti-automorphism of $\mathcal{U}(\mathfrak{k})$. The total tensor product $V^\bullet \otimes_{\mathfrak{k}} \mathcal{N}(\mathfrak{k})$ has in degree k the product

$$(V^\bullet \otimes_{\mathfrak{k}} \mathcal{N}(\mathfrak{k}))^k = \prod_{p\in\mathbb{Z}} V^p \otimes_{\mathfrak{k}} \mathcal{N}(\mathfrak{k})^{k-p}$$

and its differential is the usual differential of a double complex

$$d^k(v\otimes n) = d_V^p v \otimes n + (-1)^p v \otimes d_N^{k-p} n,$$

for $v \otimes n \in V^p \otimes \mathcal{N}(\mathfrak{k})^{k-p}$. The algebra $\mathcal{A}$ acts on $V^\bullet \otimes_{\mathfrak{k}} \mathcal{N}(\mathfrak{k})$ by

$$\pi(a)(v\otimes n) = \pi_V(a)v \otimes n,$$

for every $a \in \mathcal{A}$ and $v \otimes n \in V^\bullet \otimes_{\mathfrak{k}} \mathcal{N}(\mathfrak{k})$. The group K acts diagonally as

$$\nu(k)(v\otimes n) = \nu_V(k)v \otimes \nu_N(k)n$$

for every $k \in K$. Therefore, for every $\xi \in \mathfrak{k}$ we have $\omega(\xi)(v\otimes n) = v \otimes \omega_N(\xi)n$. Define the homotopy of actions i to be, up to a sign, the same as for $\mathcal{N}(\mathfrak{k})$. Specifically, for $v\otimes n \in V^p \otimes_{\mathfrak{k}} \mathcal{N}(\mathfrak{k})^{k-p}$ and $\xi \in \mathfrak{k}$, let $i_\xi(v\otimes n) = (-1)^p v \otimes i_\xi(n)$.

Proposition 3.5. *With the above actions and i, the total tensor product $V^\bullet \otimes_{\mathfrak{k}} \mathcal{N}(\mathfrak{k})$ is an equivariant $(\mathcal{A}, K)$-complex.*

On morphisms, tensoring with $\mathcal{N}(\mathfrak{k})$ sends f to $f \otimes 1$ for every $v \otimes n \in V^\bullet \otimes \mathcal{N}(\mathfrak{k})$. One can check this defines a chain morphism. The construction is the same if we replace $\mathcal{N}(\mathfrak{k})$ by any equivariant $(\mathcal{U}(\mathfrak{k}), K)$-complex, so we have in fact proved the following general theorem.

Theorem 3.6. *Let $V^\bullet \in \mathcal{C}_w(\mathcal{A}, K)$ and $W^\bullet \in \mathcal{C}(\mathcal{U}(\mathfrak{k}), K)$. Take the right $\mathcal{U}(\mathfrak{k})$-module structure on $V^\bullet$ determined by ω_V. Then $V^\bullet \otimes_{\mathfrak{k}} W^\bullet$ is in $\mathcal{C}(\mathcal{A}, K)$ with actions defined by*

$$\begin{array}{rcll} \pi(a)(v\otimes w) &=& \pi_V(a)v\otimes w & \forall\, a \in \mathcal{A} \text{ and} \\ \nu(k)(v\otimes w) &=& \nu_V(k)v \otimes \nu_W(k)w & \forall\, k \in K \end{array}$$

for all $v \otimes w \in V^p \otimes W^{k-p}$ and homotopies $i_\xi(v\otimes w) = (-1)^{p-1} v \otimes i_\xi w$ for all $\xi \in \mathfrak{k}$.

Define $\mathrm{Ind}'_{\mathrm{h}}(\text{ - }) := \text{ - } \otimes_{\mathcal{U}(\mathfrak{k})} \mathcal{N}(\mathfrak{k})[-d_K]$ as a functor from $\mathcal{C}_w(\mathcal{A}, K)$ to $\mathcal{C}(\mathcal{A}, K)$.

Proposition 3.7. *The functor $\mathrm{Ind}'_{\mathrm{h}}$ is naturally isomorphic to $\mathrm{Ind_h}$.*

Proof. Since $\mathfrak{k}$ is reductive, $\wedge^{d_K}\mathfrak{k} = \mathbb{C}$, the trivial representation of $\mathfrak{k}$. Therefore, the natural pairing

$$\wedge^p \mathfrak{k} \times \wedge^{d_K - p}\mathfrak{k} \to \wedge^{d_K}\mathfrak{k}$$

determines an isomorphism $\wedge^{d_K-p}\mathfrak{k} \tilde{\to} (\wedge^p\mathfrak{k})^*$ of $\mathfrak{k}$-modules. For $\tau \in \wedge^{d_K-p}\mathfrak{k}$, let $\tau^* \in (\wedge^p\mathfrak{k})^*$ denote the linear map $\tau^*(\text{ - }) = \text{ - } \wedge \tau$. Then, for any $k \in \mathbb{Z}$, this extends to a vector space isomorphism

$$\Phi^k : (V^\bullet \otimes \mathcal{N}(\mathfrak{k}))^{k-d_K} \tilde{\to} \mathrm{Hom}^k(\mathcal{N}(\mathfrak{k}), V^\bullet).$$

Clearly, the $(\mathcal{A}, K)$-module structure commutes with the vector space isomorphism. Note also, since $(\xi \wedge \tau)^* = (-1)^{p+1}\tau^* \circ i_\xi$, the homotopy i_ξ commutes with the isomorphism for all $\xi \in \mathfrak{k}$. That is, for $v \otimes \tau \in V^{k-p} \otimes \wedge^{d_K-p}\mathfrak{k}$, we have

$$\Phi^k(i_\xi(v\otimes\tau)) = i_\xi(v \otimes \tau^*).$$

The isomorphism Φ also commutes with differentials. Observe for all $(v_{k-p} \otimes \tau_{d_K-p}) \in \prod_p V^{k-p} \otimes \wedge^{d_K-p}\mathfrak{k}$, the compositions $d^k\Phi^k(v_{k-p}\otimes\tau_{d_K-p})$ and $\Phi^{k+1}d^{k-d_K}(v_{k-p}\otimes\tau_{d_K-p})$ are equal if and only if

$$(-1)^p (d_N^{d_K-(p-1)}\tau_{d_K-(p-1)})^* = \tau^*_{d_K-(p-1)} \circ d_N^p$$

for all p. Expand $\tau = \tau_{d_K-(p-1)} = \xi_1 \wedge \ldots \wedge \xi_{d_K-(p-1)}$ and let $\tau^\vee = \zeta_1 \wedge \ldots \wedge \zeta_{p-1}$ be its complement. For $\zeta \wedge d\tau$ and $d\zeta \wedge \tau$ to be nonzero, (up to sign) either $\zeta = \tau^\vee \wedge \xi_i$ for some $1 \le i \le d_K - (p-1)$ or

$\zeta = \zeta_1 \wedge \ldots \hat{\zeta_s} \ldots \wedge \zeta_{p-1} \wedge \xi_i \wedge \xi_j$ for some $1 \leq i < j \leq d_K - (p-1)$ and $1 \leq s \leq p-1$. One can verify explicitly that in either of these cases, we do in fact have the equality

$$(-1)^p (d_N^{d_K-(p-1)}\tau)^*(\zeta) = \tau^*(d_N^p(\zeta)).$$

Therefore, for any $V^\bullet \in \mathcal{C}_w(\mathcal{A}, K)$, there is an isomorphism of equivariant $(\mathcal{A}, K)$-complexes $\mathrm{Hom}^\bullet(\mathcal{N}(\mathfrak{k}), V^\bullet) \simeq V^\bullet \otimes_{\mathfrak{k}} \mathcal{N}(\mathfrak{k})[-d_K]$ which is transparently functorial in $V^\bullet$. □

3.4. **Equivariant Zuckerman Functor.** For $T \subset K$ a closed subgroup, there is a restriction functor Res_T^K from $\mathrm{D}(\mathcal{A}, K)$ to $\mathrm{D}(\mathcal{A}, T)$, given simply by restricting the K-action on any object to T. Pandžić gives an explicit construction of the right adjoint to this functor in [13], which he calls the *equivariant Zuckerman functor* and denotes by $\Gamma_{K,T}^{\mathrm{equi}}$. We recall the construction here.

Take the standard complex $\mathcal{N}(\mathfrak{k})$ as an object of $\mathcal{C}(\mathfrak{k}, T)$ via the restriction functor. Let $R(K)$ be the ring of regular functions of K. Then, for any $V^\bullet \in \mathcal{C}(\mathcal{A}, T)$, we have

$$R(K) \otimes_{\mathbb{C}} V^\bullet \in \mathcal{C}(\mathfrak{k}, T)$$

with the $(\mathfrak{k}, T)$-actions, denoted by $(\lambda_{\mathfrak{k}}, \lambda_T)$ respectively. These actions are defined for all $k \in K$ and $F \in R(K) \otimes_{\mathbb{C}} V^\bullet$ by

$$\begin{array}{rcll} (\lambda_{\mathfrak{k}}(\xi)F)(k) & = & \pi_V(\xi)F(k) + L_\xi F(k) & \forall \xi \in \mathfrak{k}, \\ (\lambda_T(t)F)(k) & = & \nu_V(t)F(kt) & \forall t \in T, \end{array}$$

where (π_V, ν_V) denote the $(\mathfrak{k}, T)$-actions on $V^\bullet$, and the $\mathfrak{k}$-action is extended to $\mathcal{U}(\mathfrak{k})$. The homotopy i is that for $V^\bullet$. There is a commuting $(\mathcal{A}, K)$-action on $R(K) \otimes_{\mathbb{C}} V^\bullet$, denoted by (π_Γ, ν_Γ) and defined for all $F \in R(K) \otimes V^\bullet$ and $k \in K$ by

$$\begin{array}{rcll} (\pi_\Gamma(a)F)(k) & = & \pi_V(\phi(k)a)F(k), & \forall a \in \mathcal{A} \\ (\nu_\Gamma(h)F)(k) & = & F(h^{-1}k) & \forall h \in K. \end{array}$$

Define the equivariant Zuckerman functor on an object $V^\bullet \in \mathcal{C}(\mathcal{A}, T)$ to be

$$\Gamma_{K,T}^{\mathrm{equi}}(V^\bullet) = \mathrm{Hom}^\bullet(\mathcal{N}(\mathfrak{k}), R(K) \otimes V^\bullet)^{(\mathfrak{k},T)},$$

with the Hom-complex taken in $\mathcal{C}_w(\mathcal{A}, K)$, then taking $(\mathfrak{k}, T)$-invariants. The $(\mathcal{A}, K)$-action on $\Gamma_{K,T}^{\mathrm{equi}}(V^\bullet)$ is denoted by (π, ν) and defined for all $f \in \Gamma_{K,T}^{\mathrm{equi}}(V^\bullet)$, $n \in \mathcal{N}(\mathfrak{k})$, and $k \in K$ as

$$\begin{array}{rcll} (\pi(a)f)(n)(k) & = & (\pi_\Gamma(a)f(n))(k) = \pi_V(\phi(k)a)f(n)(k), & \forall a \in \mathcal{A} \\ (\nu(h)f)(n)(k) & = & (\nu_\Gamma(h)f(\nu_N(h^{-1})n))(k) = f(\nu_N(h^{-1})n)(h^{-1}k) & \forall k' \in K. \end{array} \tag{4}$$

The homotopy i_ξ acts on a morphism f in degree ℓ by

$$(i_\xi f)(n)(k) = (-1)^{\ell+1} f(i_\xi n)(k)$$

for every $n \in \mathcal{N}(\mathfrak{k})$, as in the definition of $\mathrm{Ind}_{\mathrm{h}}$.

4. Equivariant Harish-Chandra Sheaves

The main technical construction required for the proof of Theorem 1.2. is that of the *geometric Zuckerman functor*. This is the localization of the equivariant Zuckerman functor to the derived equivariant $\mathscr{D}$-module categories on generalized flag varieties. In this section, we define our categories of interest and construct the geometric Zuckerman functor from the basic $\mathscr{D}$-module functors of §2.

The category of (left) G-equivariant $\mathscr{O}_X$-modules is denoted by $\mathcal{M}_G(X)$. The following theorem is well known:

Theorem 4.1. *If G acts on X freely, there is an equivalence of categories $\mathcal{M}_G(X) \simeq \mathcal{M}(X/G)$.*

For homogeneous spaces, we can make a stronger statement. Let B be a complex linear group, let $\mathcal{R}ep(B)$ be the category of algebraic representations of B.

Theorem 4.2. *If $X = G/B$, there is an equivalence of categories $\mathcal{M}_G(X) \simeq \mathcal{R}ep(B)$.*

4.1. **Group Actions on Sheaves.** Let $e_k : X \to K \times X$ be the map sending $x \mapsto (k,x)$. Then if $\mathscr{F}$ is K-equivariant with structure isomorphism $\phi : \mu^*\mathscr{F} \tilde{\to} \pi^*\mathscr{F}$, pulling back along e_k induces an isomorphism $e_k^*(\phi) : s_k^*\mathscr{F} \tilde{\to} \mathscr{F}$, where s_k is the automorphism of X given by $x \mapsto kx$. In this way, we map $k \in K$ to the isomorphism

$$e_k^*(\phi) \in \prod_{k \in K} \mathrm{Isom}(s_k^*(\mathscr{F}), \mathscr{F}),$$

and thus obtain an action of K on sections. For a local section $f \in \mathscr{F}$, define the action of $k \in K$ on f to be the local section of $\mathscr{F}$ determined by $\nu(k)f := e_k^*(\phi)^{-1}(f)$. One differentiates this action to obtain a corresponding Lie algebra action of $\mathfrak{k}$ on the sheaf $\mathscr{F}$. The general construction for Lie algebra actions on sheaves is given in [2].

4.2. **Harish-Chandra Sheaves.** Let $(\mathfrak{g}, K)$ be a Harish-Chandra pair, let X be a generalized flag variety for $\mathfrak{g}$, and let $\mathscr{D}_\lambda$ be a homogeneous twisted sheaf of differential operators on X. A *weak Harish-Chandra sheaf* for the pair $(\mathscr{D}_\lambda, K)$ is a quasi-coherent $\mathscr{D}_\lambda$-module $\mathscr{V}$ with a K-equivariant $\mathscr{O}_X$-module structure such that the action of $\mathscr{D}_\lambda$ is K-equivariant. A weak Harish-Chandra sheaf is a *Harish-Chandra sheaf* if additionally the differential of the K-action on $\mathscr{V}$ agrees with the action of $\mathfrak{k}$ induced by $\mathscr{D}_\lambda$.

A morphism of weak Harish-Chandra sheaves is a $\mathscr{D}_\lambda$-module homomorphism which respects the underlying K-equivariant structure. As with weakly equivariant Harish-Chandra modules, we will use $\mathcal{M}_w(\mathscr{D}_\lambda, K)$ to denote the category of weak Harish-Chandra sheaves and $\mathcal{M}(\mathscr{D}_\lambda, K)$ for the category of Harish-Chandra sheaves. There is an equivalence of categories for λ anti-dominant and regular

$$\mathcal{M}_{(w)}(\mathscr{D}_\lambda, K) \underset{\Delta_\lambda}{\overset{\Gamma}{\rightleftarrows}} \mathcal{M}_{(w)}(\Gamma(X, \mathscr{D}_\lambda), K).$$

We construct the derived equivariant Harish-Chandra sheaf category in the same way as the derived equivariant Harish-Chandra module category.

Definition 4.3. *An* equivariant Harish-Chandra sheaf *is a pair $(\mathscr{V}^\bullet, i)$ with $\mathscr{V}^\bullet$ a complex of weak Harish-Chandra sheaves, and i a linear map from $\mathfrak{k}$ to graded linear degree -1 endomorphisms of $\mathscr{V}^\bullet$ satisfying:*

(1) *The i_ξ are $\mathscr{D}_\lambda$-morphisms for all $\xi \in \mathfrak{k}$.*
(2) *The map $i : \mathfrak{k} \to \mathrm{Hom}_{\mathscr{D}_\lambda}(\mathscr{V}^\bullet, \mathscr{V}^\bullet[-1])$ is K-equivariant; that is, for all $k \in K$,*

$$\nu(k) \circ i_\xi \circ \nu(k^{-1}) = i_{\mathrm{Ad}(k)\xi}.$$

(3) *For all $\eta, \xi \in \mathfrak{k}$, the sum $i_\xi i_\eta + i_\eta i_\xi$ vanishes.*
(4) *There is the equality $di_\xi + i_\xi d = \omega(\xi)$, where $\omega = \nu - \pi$ and π is the action of $\mathfrak{k}$ induced from $\mathscr{D}_\lambda$.*

Define the Hom-complex $\mathrm{Hom}^\bullet(\mathscr{V}^\bullet, \mathscr{W}^\bullet)$ in the same way as for equivariant Harish-Chandra complexes. The category of equivariant Harish-Chandra sheaves $\mathcal{C}(\mathscr{D}_\lambda, K)$ has equivariant Harish-Chandra sheaves as objects and for any $\mathscr{V}^\bullet$ and $\mathscr{W}^\bullet$ the morphisms between them are $\mathrm{Hom}^0(\mathscr{V}^\bullet, \mathscr{W}^\bullet)$. The homotopy category of equivariant Harish-Chandra sheaves $\mathcal{K}(\mathscr{D}_\lambda, K)$ has the same objects, but the zeroth cohomology $\mathrm{H}^0(\mathrm{Hom}^\bullet(\mathscr{V}^\bullet, \mathscr{W}^\bullet))$ of the Hom-complex from $\mathscr{V}^\bullet$ to $\mathscr{W}^\bullet$. The derived equivariant Harish-Chandra sheaf category $\mathrm{D}(\mathscr{D}_\lambda, K)$ is the localization of $\mathcal{K}(\mathscr{D}_\lambda, K)$ with respect to quasi-isomorphisms. According to [12], when X is the full flag variety and λ is regular, the derived global sections functor

$$\mathbf{R}\Gamma : \mathrm{D}^+(\mathscr{D}_\lambda, K) \to \mathrm{D}^+(\mathcal{U}_{[\lambda]}, K),$$

is an equivalence. This equivariant form of Beilinson-Bernstein localization indicates the equivariant Harish-Chandra sheaf categories are the appropriate geometric category in which to work in the context of the results of [13].

The categories of interest in the construction of the geometric Zuckerman functor are as follows. Given a generalized flag variety X for a Harish-Chandra pair $(\mathfrak{g}, K)$ and a homogeneous tdo $\mathscr{D}_\lambda$ on X, we have the categories:

(1) The abelian category $\mathcal{M}_{(w)}(\mathscr{D}_\lambda, K)$ of (weak) Harish-Chandra sheaves on X for the pair $(\mathscr{D}_\lambda, K)$.

(2) The category $\mathcal{C}(\mathcal{M}_{(w)}(\mathscr{D}_\lambda, K))$ of complexes of (weak) Harish-Chandra sheaves on X, and its homotopy category $\mathcal{K}(\mathcal{M}_{(w)}(\mathscr{D}_\lambda, K))$ and derived category $\mathrm{D}(\mathcal{M}_{(w)}(\mathscr{D}_\lambda, K))$. For concision, in the case of weak Harish-Chandra sheaves we may abbreviate the above notation by

$$(*)_w(\mathscr{D}_\lambda, K) = (*)(\mathcal{M}_w(\mathscr{D}_\lambda, K))$$

for $(*) = \mathcal{C}$, $\mathcal{K}$, or D.

(3) The category of equivariant Harish-Chandra sheaves for $(\mathscr{D}_\lambda, K)$, denoted $\mathcal{C}(\mathscr{D}_\lambda, K)$ and the homotopy and derived categories $(*)(\mathscr{D}_\lambda, K)$ with $(*) = \mathcal{K}$ and D, respectively.

(4) For $S \subseteq K$ a closed subgroup, the categories $(*)(\mathcal{M}(\mathscr{D}_\lambda, K), S^{(w)})$, where $(*) = \mathcal{C}, \mathcal{K}$ or D, with objects consisting of complexes of Harish-Chandra modules for $(\mathscr{D}_\lambda, K)$ that are (weakly) S-equivariant complexes. With respect to the notation of this list, we have

$$(*)(\mathcal{M}(\mathscr{D}_\lambda, K), S^w) = (*)(\mathcal{M}(\mathscr{D}_\lambda, K \times S^w)).$$

4.3. **The Geometric** Ind_h**.** The forgetful functor

$$\mathrm{For}_\mathrm{h} : \mathcal{C}(\mathscr{D}_\lambda, K) \to \mathcal{C}_w(\mathscr{D}_\lambda, K)$$

forgets the homotopy i, as did the forgetful functor for the equivariant $(\mathcal{A}, K)$-complexes of §3.2.

Proposition 4.4. *The forgetful functor* For_h *has a right adjoint.*

We will now construct a functor $\mathrm{Ind}_\mathrm{h} : \mathcal{C}_w(\mathscr{D}_\lambda, K) \to \mathcal{C}(\mathscr{D}_\lambda, K)$ and show it is the right adjoint to For_h in Proposition 4.6. For an object $\mathscr{V}^\bullet \in \mathcal{C}(\mathscr{D}_\lambda, K)$, put

$$\mathrm{Ind}_\mathrm{h}(\mathscr{V}^\bullet) = \mathrm{Hom}^\bullet_{\mathcal{U}(\mathfrak{k})}(\mathcal{N}(\mathfrak{k}), \mathscr{V}^\bullet),$$

where $\mathcal{N}(\mathfrak{k})$ is the standard complex thought of as a constant sheaf on X. For any $\xi \in \mathfrak{k}$ define

$$i_\xi : \mathrm{Ind}_\mathrm{h}(\mathscr{V}^\bullet) \to \mathrm{Ind}_\mathrm{h}(\mathscr{V}^\bullet)[-1], \quad i_\xi f(u \otimes \tau) = (-1)^j f(i_\xi(u \otimes \tau))$$

for f in degree j, where $i_\xi : \mathcal{N}(\mathfrak{k}) \to \mathcal{N}(\mathfrak{k})[-1]$ is the map

$$i_\xi(u \otimes \tau) = -u \otimes \xi \wedge \tau.$$

As a $(\mathscr{D}_\lambda, K)$-module, $\mathscr{D}_\lambda$ acts on $f \in \mathrm{Ind}_\mathrm{h}(\mathscr{V}^\bullet)$ by its action on $\mathscr{V}^\bullet$, and K acts by conjugating f by the K-actions on each factor. That is, for all local sections $T \in \mathscr{D}_\lambda$, we have $(Tf)(u \otimes \tau) = Tf(u \otimes \tau)$ and

$$(\nu(k)f)(u \otimes \tau) = \nu_V(k) f(\nu_N(k^{-1})u \otimes \tau),$$

where ν_V is the K-action on local sections of $\mathscr{V}^\bullet$ and ν_N the K-action on $\mathcal{N}(\mathfrak{k})$.

Proposition 4.5. *The pair* $(\mathrm{Ind}_\mathrm{h}(\mathscr{V}^\bullet), i)$ *is an object of* $\mathcal{C}(\mathscr{D}_\lambda, K)$ *for all* $\mathscr{V}^\bullet \in \mathcal{C}_w(\mathscr{D}_\lambda, K)$.

We omit the proof which consists entirely of computations following the definitions.

Proposition 4.6. *The functor* Ind_h *is right adjoint to* For_h.

We again omit the completely computational proof, which can be found in [7, §5.3]. A version of this proposition is also presented without proof in [3, Prop. 2.13.2]. We use the Hom-complex in the proof of adjointness, which allows us to conclude that Ind_h is also the right adjoint to For_h for the homotopy category. Moreover, the functor Ind_h preserves $\mathcal{K}$-injective objects. The proposition below follows immediately from Lemma 4.8 below.

Proposition 4.7. *If the category* $\mathcal{C}_w(\mathscr{D}_\lambda, K)$ *has enough* $\mathcal{K}$*-injectives, then the categories* $(*)(\mathscr{D}_\lambda, K)$ *for* $(*) = \mathcal{C}, \mathcal{K}$ *or* D *have enough* $\mathcal{K}$*-injectives.*

Lemma 4.8. *If* $f : \mathscr{V}^\bullet \to \mathscr{I}^\bullet$ *is a quasi-isomorphism of* $\mathscr{V}^\bullet$ *with a* $\mathcal{K}$*-injective* $\mathscr{I}^\bullet$*, then* $\phi_f := \mathrm{Ind}_\mathrm{h}(f)$ *is also a quasi-isomorphism.*

Proof. Recall from [14, Prop. 1.5] that $\mathscr{I}^\bullet$ is $\mathcal{K}$-injective if and only if for all diagrams

$$\begin{tikzcd} \mathscr{V}^\bullet \arrow[r, "\phi"] \arrow[d, "s"'] & \mathscr{I}^\bullet \\ \mathscr{W}^\bullet & \end{tikzcd}$$

with s a quasi-isomorphism, there exists a unique morphism $g_\phi : \mathscr{W}^\bullet \to \mathscr{I}^\bullet$ such that $[g_\phi s] = [\phi]$ in the homotopy category.

Consider the diagram

$$\begin{tikzcd} \mathcal{N}(\mathfrak{k}) \arrow[r, "\psi"] \arrow[d, "s"'] & \mathscr{I}^\bullet[k] \\ \mathbb{C} & \end{tikzcd}$$

for any k, where s is the usual quasi-isomorphism. Then, the unique g_ψ which exists such that $[g_\psi s] = [\psi]$ must send 1 to $\psi^0(1)$.

Next, let $[v] \in \mathrm{H}^k(\mathscr{V}^\bullet)$ such that $[f(v)] = [\psi^0(1)] \in \mathrm{H}^k(\mathscr{W}^\bullet)$. Since f is a quasi-isomorphism, such $[v]$ is unique. Thus there exists a unique $g_f : \mathbb{C} \to \mathscr{I}^\bullet$ such that $[g_f s] = [\phi_f(v)]$. We have

$$[g_f(1)] = [f(v)] = [g_\psi(1)] \in \mathrm{H}^k(\mathscr{W}).$$

This implies $[g_f] = [g_\psi]$, and consequently $[\phi_f(v)] = [\psi]$. Moreover, this result verifies both injectivity and surjectivity of the morphism

$$[\phi_f] : \mathrm{H}^k(\mathscr{V}^\bullet) \to \mathrm{H}^k(\mathrm{Ind_h}(\mathscr{I}^\bullet))$$

for all k. □

4.4. **Reduction Principle.** Although the category $\mathrm{D}(\mathscr{D}_\lambda, K)$ is not derived from its heart $\mathcal{M}(\mathscr{D}_\lambda, K)$, we define in this section a notion of a derived functor. Trivially, any exact functor on $\mathcal{M}_w(\mathscr{D}_\lambda, K)$ extends to a functor on $\mathrm{D}(\mathscr{D}_\lambda, K)$. Moreover, the forgetful functor from $\mathcal{M}(\mathscr{D}_\lambda, K)$ to $\mathcal{M}_w(\mathscr{D}_\lambda, K)$ and $\mathcal{C}(\mathscr{D}_\lambda, K)$ to $\mathcal{C}_w(\mathscr{D}_\lambda, K)$ are obviously faithful, so any functor on the weakly equivariant categories lifts immediately to Harish-Chandra sheaves and equivariant Harish-Chandra complexes. The lifting of properties such as exactness, adjointness, etc. follows trivially. The forgetful functors at the homotopy and derived level are not necessarily faithful, since homotopies of morphisms in the equivariant Harish-Chandra categories must anti-commute with the additional structure map i with which each object is equipped. In this case, the lifting of functors from the weakly equivariant categories to the equivariant categories is non-trivial, but made possible by the existence of the right adjoint $\mathrm{Ind_h}$.

Since $\mathcal{C}_w(\mathscr{D}_\lambda, K)$ has enough $\mathcal{K}$-injectives, so does $\mathcal{C}(\mathscr{D}_\lambda, K)$ and the corresponding homotopy and derived categories. In this case, any left exact functor on $\mathcal{M}_w(\mathscr{D}_\lambda, K)$ defines a right derived functor on $\mathrm{D}^*(\mathscr{D}_\lambda, K)$, up to imposing appropriate finiteness conditions (depending on $*$) for cohomological dimension. In the next sections, the construction of the functors needed for the main theorem are given in terms of left exact functors on $\mathcal{M}_w(\mathscr{D}_\lambda, K)$ with finite right cohomological dimension.

4.5. **Restriction of Group Actions.** Let $S \subset K$ be a closed subgroup. There is a restriction functor Res_S^K from $\mathcal{M}(\mathscr{D}_\lambda, K)$ to $\mathcal{M}(\mathscr{D}_\lambda, S)$ defined by restricting the K-action to S. Since the K-action on an equivariant sheaf $\mathscr{V}$ is defined by an isomorphism $\phi : \mu^*\mathscr{V} \tilde{\to} \pi^*\mathscr{V}$, with $\mu, \pi : K \times X \to X$ the usual action and projection morphisms (respectively), the restriction of the K-action comes from taking ϕ to $j^*\phi$, where $j : S \times X \to K \times X$ is the obvious inclusion. The restriction functor is exact and therefore extends to the equivariant derived categories.

Definition 4.9. *For $S \subset K$ a closed subgroup define the restriction functor*

$$\mathrm{Res}_S^K : \mathcal{C}(\mathscr{D}_\lambda, K) \to \mathcal{C}(\mathscr{D}_\lambda, S)$$

by restricting the K-action to S and the map i to $\mathfrak{s}$.

Let π_*^K be the direct image functor π_* composed with the functor of K-invariant sections $(\text{ - })^K$.

Proposition 4.10. *There is a natural isomorphism* $\mathrm{Res}_S^K \simeq \pi_*^K \mu^*$.

Proof. It is enough to prove the proposition for $\mathcal{M}(\mathscr{D}_\lambda, K)$, since μ^* and π_*^K are exact. Let $\mathscr{V} \in \mathcal{M}_w(\mathscr{D}_\lambda, K)$. There is an isomorphism $\phi : \mu^*\mathscr{V} \tilde{\to} \pi^*\mathscr{V}$ defining the K-action. Additionally, the functor π_*^K is the inverse to π^* in the equivalence

$$\mathcal{M}(\mathscr{O}_X, S) \simeq \mathcal{M}(\mathscr{O}_{K\times X}, K \times S).$$

Therefore, the equivalence ϕ pushes down to an isomorphism $\pi_*^K(\phi)$ from $\pi_*^K\mu^*\mathscr{V}$ to $\mathrm{Res}_S^K\mathscr{V}$. Let $\pi_S, \mu_S : S \times X \to X$ be the projection and action morphisms, and let $j : S \times X \to K \times X$ be induced from the inclusion $S \to K$. Then we have $\mu_S = \mu \circ j$ and likewise $\pi_S = \pi \circ j$. Moreover, there is a shear morphism $s : K \times X \to K \times X$ such that we have $\mu = \pi \circ s$. Consequently,we have $\pi_S^*\pi_*^K = j^*$. Thus the isomorphisms $\pi_S^*\pi_*^K(\phi)$ and $j^*(\phi)$ from $\mu_S^*\mathscr{V}$ to $\pi_S^*\mathscr{V}$ are equal.

Now, let $f : \mathscr{V} \to \mathscr{W}$ be any morphism in $\mathcal{M}_w(\mathscr{D}_\lambda, K)$. It restricts to a morphism in $\mathcal{M}(\mathscr{D}_\lambda, S^w)$. Then, the equalities $f \circ \pi_*^K(\phi) = \pi_*^K(\pi^*(f) \circ \phi) = \pi_*^K(\phi) \circ \pi_*^K\mu^*(f)$ complete the proof. □

4.6. **The Geometric Zuckerman Functor.** As in the algebraic setting, the geometric restriction functor Res_S^K has a right adjoint. We will construct the geometric Zuckerman functor $\Gamma_{K,S}^{\mathrm{geo}}$ and prove that it is the right adjoint to Res_S^K.

Let X be a K-variety and S a closed subgroup of K. Lemma 1.8.6 of [2] states there is an equivalence

$$\mathcal{M}_w(\mathscr{D}_\lambda, K) \simeq \mathcal{M}(\mathscr{D}_\lambda \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{k}), K).$$

Also in [2], given a free K-action on $K \times X$, let $q : K \times X \to X$ be the quotient map. There is then a pair of equivalences

$$\mathcal{M}_w(\mathscr{D}_\lambda^q, K) \underset{q^\circ}{\overset{q_*^K}{\rightleftarrows}} \mathcal{M}(\mathscr{D}_\lambda \otimes \mathcal{U}(\mathfrak{k})) \quad \text{and} \tag{5}$$

$$\mathcal{M}(\mathscr{D}_\lambda^q, K) \underset{q^\circ}{\overset{q_*^K}{\rightleftarrows}} \mathcal{M}(\mathscr{D}_\lambda). \tag{6}$$

The equivalence (5) generates an equivalence for weakly equivariant complexes

$$\mathcal{C}_w(\mathscr{D}_\lambda^q, K) \underset{q^\circ}{\overset{q_*^K}{\rightleftarrows}} \mathcal{C}(\mathscr{D}_\lambda^q \otimes \mathcal{U}(\mathfrak{k})).$$

We can naturally extend these equivalences to include (weakly) S-equivariant sheaves whenever q is a S-equivariant morphism and the S-action on $K \times X$ commutes with the K-action. Let π and $\mu : K \times X \to X$ be the usual projection and action maps, respectively. The product $K \times S$ acts on $K \times X$ by $(k', s)(k, x) = (k'ks^{-1}, sx)$ for all $(k, x) \in K \times X$ and $(k', s) \in K \times S$. There is also a $K \times S$-action on X with K acting trivially and S acting by the restriction of the original K-action. With these actions on $K \times X$ and X, the map π is $K \times S$-equivariant. Similarly, for the $K \times S$-action on X given by the μ-action of K and the trivial S-action, the morphism μ is $K \times S$-equivariant. If S is trivial, we can recover the situation of [2] described above by letting $q = \pi$ or μ.

The S-equivariance of π and the fact that the inverse image π° is an exact functor from $\mathcal{M}(\mathscr{D}_\lambda, S^{(w)})$ to $\mathcal{M}(\mathscr{D}_\lambda^\pi, K \times S^{(w)})$ imply that π° extends to a functor of derived equivariant categories

$$\pi^\circ : \mathrm{D}(\mathscr{D}_\lambda, S^{(w)}) \to \mathrm{D}(\mathcal{M}(\mathscr{D}_\lambda^\pi, K), S^{(w)}).$$

Since $\mathscr{D}_\lambda$ is a K-equivariant $\mathscr{O}_X$-module on X, there is a canonical isomorphism between $\mu^*\mathscr{D}_\lambda$ and $\pi^*\mathscr{D}_\lambda$. Consequently, we have an induced isomorphism $\mathscr{D}_\lambda^\mu \simeq \mathscr{D}_\lambda^\pi$ of their respective sheaves of differential endomorphisms. Therefore, there is a natural isomorphism of categories $\mathcal{M}(\mathscr{D}_\lambda^\mu) = \mathcal{M}(\mathscr{D}_\lambda^\pi)$.

We use the above equivalences to motivate our construction of the geometric Zuckerman functor. The direct image μ_*^K above does not land in the category of strongly K-equivariant sheaves, since it is not clear what the $\mathscr{D}$-module structure is away from the K-equivariant sections. The $\mathscr{D}$-module direct image μ_+ corrects for this problem, as we show in the proposition below.

Proposition 4.11. *The $\mathscr{D}$-module direct image functor μ_+ takes $\mathcal{M}(\mathscr{D}^{\mu}_{\lambda}, K\times S^{(w)})$ to $\mathcal{C}^{\mathrm{b}}(\mathcal{M}(\mathscr{D}_{\lambda}, S^{(w)}), K)$ and $\mathcal{C}^{\mathrm{b}}(\mathcal{M}(\mathscr{D}^{\mu}_{\lambda}, K), S^{(w)})$ to $\mathcal{C}^{\mathrm{b}}(\mathscr{D}_{\lambda}, K\times S^{(w)})$.*

Proof. We need only construct a homotopy of the $\mathfrak{k}$-actions

$$i : \mathfrak{k} \to \mathrm{Hom}_{\mathscr{D}_{\lambda}}(\mu_+\mathscr{V}, \mu_+\mathscr{V}[-1])$$

for every object $\mathscr{V} \in \mathcal{M}(\mathscr{D}^{\mu}_{\lambda}, K\times S^{(w)})$. Since μ is a surjective submersion, the direct image functor μ_+ is equal to $\mu_*(\text{ - } \otimes_{\mathscr{O}_{K\times X}} \mathcal{T}^{\bullet}_{K\times X/X})$. The sheaf $\mathcal{T}_{K\times X}$ equals the exterior tensor product $\mathcal{T}_K \boxtimes \mathcal{T}_X$, and therefore the relative sheaf of differentials

$$\mathcal{T}_{K\times X/X} \simeq \pi_K^*\mathcal{T}_K.$$

Moreover, K is affine so $\mathcal{T}_K = \mathscr{O}_K \otimes_{\mathbb{C}} \mathfrak{k}$ and hence the inverse image is $\pi_K^*\mathcal{T}_K^{\bullet} = \mathscr{O}_{K\times X}\otimes_{\mathbb{C}} \wedge^{\bullet}\mathfrak{k}$. Namely, we have equalities

$$\mu_+(\text{ - }) = \mu_*(\text{ - } \otimes_{\mathbb{C}} \wedge^{\bullet}\mathfrak{k}) = \mu_*(\text{ - }) \otimes_{\mathcal{U}(\mathfrak{k})} \mathcal{N}(\mathfrak{k}).$$

Define the homotopy map i to be that coming from $\mathcal{N}(\mathfrak{k})$. □

Define the functor of K-invariant sections on $\mathcal{C}_{(w)}(\mathscr{D}_{\lambda}, K)$ by

$$(\text{ - })^K = (\text{ - })^{(K,\mathcal{N})} : \mathcal{C}_{(w)}(\mathscr{D}_{\lambda}, K) \to \mathcal{C}(\mathscr{D}_{\lambda}),$$

where $\mathcal{N} = \mathcal{U}(\mathfrak{k})$ in the case of weakly equivariant complexes, and $\mathcal{N}(\mathfrak{k})$ for equivariant complexes. By $\mathcal{N}(\mathfrak{k})$-invariants, we mean $\mathcal{U}(\mathfrak{k})$-invariants which are also invariant for the homotopy map i. The invariants functor $(\text{ - })^K$ is right adjoint to the trivial inclusion $\mathrm{Triv}_{(w)}$ from $\mathcal{C}(\mathscr{D}_{\lambda})$ into $\mathcal{C}_{(w)}(\mathscr{D}_{\lambda}, K)$.

Definition 4.12. *For $(*) = \mathcal{C}$ or $\mathcal{K}$, define the geometric Zuckerman functor from $(*)^{\mathrm{b}}(\mathscr{D}_{\lambda}, S)$ to $(*)^{\mathrm{b}}(\mathscr{D}_{\lambda}, K)$ to be*

$$\Gamma^{\mathrm{geo}}_{K,S} := \mu^S_+\pi^{\circ}[-d_K].$$

By the reduction principle, the geometric Zuckerman functor $\Gamma^{\mathrm{geo}}_{K,S}$ is also defined for derived equivariant categories.

Proposition 4.13. *The geometric Zuckerman functor $\Gamma^{\mathrm{geo}}_{K,S}$ is right adjoint to Res^K_S.*

Proof. Recall there is a natural isomorphism

$$\mathrm{Res}^K_S \simeq \pi^K_*\mu^{\circ}$$

and note π^K_* is the inverse to π°. Therefore, we need only show μ° is left adjoint to $\mu^S_+[-d_K]$. Since μ is smooth, we know $\mu^{\circ} \dashv \mu^S_+[-d_K]$ is an adjoint pair when S is trivial. Additionally, Proposition 4.10. shows that $(\text{ - })^S$ is right adjoint to $\mathrm{Triv}_{(w)}$. The restriction functor Res^K_S factors through $\mathcal{C}(\mathscr{D}_{\lambda}, K\times S)$ by the functor Triv. In fact, we should have defined $\mathrm{Res}^K_S = \pi^K_*\mu^{\circ}\circ \mathrm{Triv}$, in which case it is clear that

$$\mu^{\circ}\circ \mathrm{Triv} \dashv \mu^S_+[-d_K]$$

as a composition of two adjoint pairs. □

Before moving on to examine properties of $\Gamma^{\mathrm{geo}}_{K,S}$, it will be useful to generalize the equivalences (5) and (6) to derived equivalences for equivariant complexes.

Lemma 4.14. *For K acting freely on Z and $\mathscr{D}$ a sheaf of twisted differential operators on the quotient Z/K, there is an equivalence of categories*

$$\mathcal{C}(\mathscr{D}^q, K) \underset{q^{\circ}}{\overset{q^K_+[-d_K]}{\rightleftarrows}} \mathcal{C}(\mathscr{D}).$$

Proof. By the reduction principle, the lemma follows from the equivalence

$$\mathcal{M}_w(\mathscr{D}^q, K) \underset{q^\circ}{\overset{q_*^K}{\rightleftarrows}} \mathcal{M}(\mathscr{D})$$

whenever $q_+^K[-d_K] \simeq q_*^K$. Recall there are isomorphisms $q_+^K(\mathscr{V})[-d_K] \simeq (q_*(\mathscr{V}) \otimes_{\mathcal{U}(\mathfrak{k})} \mathcal{N}(\mathfrak{k}))^K[-d_K]$ and $(q_*(\mathscr{V}) \otimes_{\mathcal{U}(\mathfrak{k})} \mathcal{N}(\mathfrak{k}))^K[-d_K] = \mathrm{Hom}_{\mathfrak{k}}(\mathcal{N}(\mathfrak{k}), q_*\mathscr{V})^K = q_*^K(\mathscr{V})$ from which the desired equivalence of abelian categories follows. □

This equivalence respects (weak) S-equivariance when the S-action commutes with K and q is S-equivariant.

4.7. **Properties of $\Gamma_{K,S}^{\mathrm{geo}}$.** In this section, we use the geometric Zuckerman functor $\Gamma_{K,S}^{\mathrm{geo}}$ to construct standard modules on generalized flag varieties. We fix the following notations. As above, let $(\mathfrak{g}, K)$ be a Harish-Chandra pair, let X be a generalized flag variety of $\mathfrak{g}$, let Q be a K-orbit on X and let $i : Q \to X$ be the inclusion. Fix a point $x \in Q$ and let $i_x : x \to Q$ and $j_x = i \circ i_x$ denote the inclusions of x to Q and X respectively. Fix a homogeneous tdo $\mathscr{D}_\lambda$ on X and let $\mathcal{D}_{[\lambda]}$ denote the global sections of $\mathscr{D}_\lambda$, with $[\lambda] \in \mathfrak{h}^*/\mathcal{W}$ the Weyl group orbit of λ. Denote the stabilizer of x in K by S_x.

Proposition 4.15. *The diagram below is commutative:*

$$\begin{tikzcd} \mathrm{D}^{\mathrm{b}}(\mathscr{D}_\lambda^i, S_x) \arrow[r, "\Gamma_{K,S_x}^{\mathrm{geo}}"] \arrow[d, "i_+"'] & \mathrm{D}^{\mathrm{b}}(\mathscr{D}_\lambda^i, K) \arrow[d, "i_+"] \\ \mathrm{D}^{\mathrm{b}}(\mathscr{D}_\lambda, S_x) \arrow[r, "\Gamma_{K,S_x}^{\mathrm{geo}}"] & \mathrm{D}^{\mathrm{b}}(\mathscr{D}_\lambda, K). \end{tikzcd}$$

Proof. Consider the diagram

$$\begin{tikzcd} K \times Q \arrow[r, "i"] \arrow[d, "\mu"'] & K \times X \arrow[d, "\mu"] \\ Q \arrow[r, "i"] & X. \end{tikzcd}$$

Then, $i_+\mu_+ = \mu_+ i_+$ and moreover, $i_+\mu_+^{S_x} = \mu_+^{S_x} i_+$ since i is S_x-equivariant. Base change for $\mathscr{D}_\lambda$-modules allows us to commute i_+ past π°. □

Proposition 4.16. *The diagram below is commutative:*

$$\begin{tikzcd} \mathrm{D}^{\mathrm{b}}(\mathscr{D}_\lambda, S_x) \arrow[r, "\Gamma_{K,S_x}^{\mathrm{geo}}"] \arrow[d, "\mathbf{R}\Gamma"'] & \mathrm{D}^{\mathrm{b}}(\mathscr{D}_\lambda, K) \arrow[d, "\mathbf{R}\Gamma"] \\ \mathrm{D}^{\mathrm{b}}(\mathcal{D}_{[\lambda]}, S_x) \arrow[r, "\Gamma_{K,S_x}^{\mathrm{equi}}"] & \mathrm{D}^{\mathrm{b}}(\mathcal{D}_{[\lambda]}, K). \end{tikzcd}$$

Proof. The reduction principle allows us to work with categories of complexes alone. It is clear from the previous constructions that for any $\mathscr{V}^\bullet \in \mathcal{C}^{\mathrm{b}}(\mathscr{D}_\lambda, S_x)$, we have

$$\mathbf{R}\Gamma(X, \Gamma_{K,S_x}^{\mathrm{geo}}\mathscr{V}^\bullet) = \mathrm{Hom}^\bullet_{(\mathfrak{k}, S_x, \mathcal{N}(\mathfrak{s}))}(\mathcal{N}(\mathfrak{k}), \mathbf{R}\Gamma(X, \mu_*\pi^\circ\mathscr{V}^\bullet))$$

as $(\mathcal{D}_{[\lambda]}, K)$-complexes. For any open set $U \subset X$, we have by definition $\mu_*\pi^\circ\mathscr{V}^\bullet(U) = R(K) \otimes \mathscr{V}^\bullet(U)$ where $R(K)$ is the ring of regular functions on K. Therefore, we have

$$\mathbf{R}\Gamma(X, \mu_*\pi^\circ\mathscr{V}^\bullet) = R(K) \otimes \mathbf{R}\Gamma(X, \mathscr{V}^\bullet).$$

Local sections of $\mu_*\pi^*\mathscr{V}^\bullet$ are functions F from K to $\mathscr{V}^\bullet$.

The shift isomorphism $\sigma : K \times X \to K \times X$ (defined by $\sigma(k, x) = (k, kx)$) relates μ and π by $\mu = \pi \circ \sigma$. Let $\phi : \mu^*\mathscr{D}_\lambda \tilde{\to} \pi^*\mathscr{D}_\lambda$ be the K-equivariance isomorphism, and let $\phi^* : \mathscr{D}_\lambda^\mu \tilde{\to} \mathscr{D}_\lambda^\pi$ be the

induced isomorphism of tdo's. The isomorphism ϕ^* locally sends a section $T \in \mathscr{D}_\lambda^\mu$ to $\phi \circ T \circ \phi^{-1}$, where $\phi : \mu^* \mathscr{D}_\lambda \tilde{\to} \pi^* \mathscr{D}_\lambda$ sends $X \in \mu^* \mathscr{D}_\lambda$ to $\phi(X) = X \circ \sigma^{-1}$. Now consider the S_x-action. Let

$$\tilde{\mu}, \tilde{\pi} : S_x \times K \times X \to K \times X$$

be the action and projection morphisms, respectively. For any $t \in S_x$, let

$$\tilde{e}_t : K \times X \to S_x \times K \times X$$

be the inclusion $(k, x) \mapsto (t, k, x)$. If ψ is the S_x-equivariance morphism for $\mathscr{V}^\bullet$, then the S_x-action on $\pi^* \mathscr{V}^\bullet$ is given by

$$\lambda_s(t)(f \otimes v) = \pi^* e_t^*(\psi)^{-1}(f \otimes v),$$

where $e_t : X \to S_x \times X$, $e_t : x \mapsto (t, x)$. Hence for a local section $F \in \mu_* \pi^* \mathscr{V}^\bullet$, we have

$$(\lambda_s(t)F)(k) = e_t^*(\psi)^{-1}(F(kt)).$$

The $\mathcal{U}(\mathfrak{k})$-module structure on $\mu_* \pi^* \mathscr{V}^\bullet$ is induced by ϕ^*. For every $F \in \mu_* \pi^* \mathscr{V}$ and $\xi \in \mathfrak{k}$, we have

$$(\lambda_{\mathfrak{k}}(\xi)F)(k) = \phi^{-1}(\xi \phi(F))(k) = L_\xi F(k) + \pi_V(\xi) F(k).$$

On the other hand, $\mathscr{D}_\lambda^\pi = \mathscr{D}_K \boxtimes \mathscr{D}_\lambda$, so for all $T \in \mathscr{D}_\lambda$ and $F \in \mu_* \pi^* \mathscr{V}^\bullet$, we have

$$(\pi_\Gamma(T)F)(k) = (\phi^{-1} \circ T \circ \phi)(F)(k) = \pi_V(\nu_D(k^{-1}) T \nu_D(k)) F(k),$$

where ν_D denotes the K-action on $\mathscr{D}_\lambda$, and π_V the $\mathscr{D}_\lambda$-action on $\mathscr{V}^\bullet$. Since $\sigma_k^* \mu_* \pi^* \mathscr{V}^\bullet = \mu_* \pi^* \mathscr{V}^\bullet$ for all $k \in K$, the K-action on $F \in \mu_* \pi^* \mathscr{V}^\bullet$ is defined by $(\nu_\Gamma(k)F)(k') = F(k^{-1}k')$. The $(\mathcal{D}_\lambda, K)$-actions on $\mathrm{Hom}^\bullet_{(\mathfrak{k}, S_x, \mathcal{N}(\mathfrak{s}_x))}(\mathcal{N}(\mathfrak{k}), \mathbf{R}\Gamma(X, \mu_* \pi^* \mathscr{V}^\bullet))$ are given as in the definition of the algebraic functor $\mathrm{Ind}_{\mathrm{h}}$. Likewise, the homotopy of actions i is defined for a morphism $f : \mathcal{N}(\mathfrak{k}) \to R(K) \otimes \mathbf{R}\Gamma(X, \mathscr{V}^\bullet)$ of degree ℓ by

$$(i_\xi f)(n)(k) = (-1)^\ell f(i_{\mathrm{Ad}(k)\xi} n)(k)$$

for all $n \in \mathcal{N}(\mathfrak{k})$, $k \in K$, where we take the $(\mathcal{U}(\mathfrak{k}), K)$-module structure on $\mathcal{N}(\mathfrak{k})$ from §3.2. Then, by Proposition 4.16. we have an isomorphism

$$\mathbf{R}\Gamma(X, \Gamma_{K, S_x}^{\mathrm{geo}}(\mathscr{V}^\bullet)) \simeq \Gamma_{K, S_x}^{\mathrm{equi}}(\mathbf{R}\Gamma(X, \mathscr{V}^\bullet))$$

of $(\mathcal{D}_{[\lambda]}, K)$-complexes. □

Proposition 4.17. *If τ is a connection on Q compatible with $\mathscr{D}^i$, then*

$$\Gamma_{K, S_x}^{\mathrm{geo}} i_{x+} T_x \tau [d_Q] \simeq \tau.$$

Proof. Denote the quotient and projection maps $q : K \to Q$ and $p : K \to x$ respectively and observe $q = \mu \circ i_x$. Let $i_x : K \to K \times Q$ be the lift of $i_x : x \to Q$ under the projection $\pi : K \times Q \to Q$. Base change for $\mathscr{D}$-modules then gives the equivalence

$$\Gamma_{K, S_x}^{\mathrm{geo}} i_{x+} [d_Q] = q_+^{S_x} \pi^\circ [-d_{S_x}].$$

Lemma 4.14. provides an equivalence of categories

$$\mathcal{C}^{\mathrm{b}}(\mathscr{D}^{i \circ q}, S_x) \underset{p_+^K q^\circ [-d_K]}{\overset{q_+^{S_x} p^\circ [-d_{S_x}]}{\rightleftarrows}} \mathcal{C}^{\mathrm{b}}(\mathscr{D}^i, K).$$

The corresponding equivalence for the underlying $\mathscr{O}$-module structure is

$$\mathcal{R}ep_{S_x} \underset{p_*^K q^*}{\overset{q_*^{S_x} p^*}{\rightleftarrows}} \mathcal{M}(\mathscr{O}_Q, K), \tag{7}$$

which exists since τ is a vector bundle. In this setting, we have $i_x^\circ \mathrm{Res}_{S_x}^K \tau = p_*^K q^* \tau = T_x \tau$. The equivalence (7) thus implies there is a natural isomorphism $\tau \simeq q_+^{S_x} \pi^\circ T_x \tau$.

□

Proposition 4.18. *With the above notation, we have $p^\circ \Gamma_{K, S_x}^{\mathrm{geo}} = \Gamma_{K, S_x}^{\mathrm{geo}} p^\circ$.*

Proof. The inverse image p° obviously commutes with π°, and it commutes with $\mu_+^{S_x}$ by base change since p is S_x-equivariant. □

5. Cohomology of Derived Standard Modules

5.1. **The Embedding Theorem.** For a Harish-Chandra pair $(\mathfrak{g}, K)$, let X_θ be a partial flag variety of type θ. Fix a tdo $\mathscr{D}_\lambda$ on X_θ. Let X denote the full flag variety of $\mathfrak{g}$. There is a natural fibration

$$p : X \to X_\theta$$

and a corresponding natural morphism from $\mathcal{U}^p_{[\lambda]} := \Gamma(X, \mathscr{D}^p_\lambda)$ to $\mathcal{D}_{[\lambda]} := \Gamma(X_\theta, \mathscr{D}_\lambda)$. We obtain a pull-back functor

$$p^* : \mathcal{M}(\mathcal{D}_{[\lambda]}) \to \mathcal{M}(\mathcal{U}^p_{[\lambda]})$$

of modules over these rings in the usual way. The pull-back p^* is related to p° by the following theorem.

Theorem 5.1 (Embedding Theorem 1.3)**.** *The inverse image functor*

$$p^\circ : \mathcal{M}(\mathscr{D}_\lambda) \to \mathcal{M}(\mathscr{D}^p_\lambda)$$

is fully faithful for all λ, and for λ anti-dominant, we have $\Gamma \circ p^\circ = p^* \circ \Gamma$.

Proof. We will prove full faithfulness of p° by constructing a functor from $\mathcal{M}(\mathscr{D}^p_\lambda) \to \mathcal{M}(\mathscr{D}_\lambda)$ which is quasi-inverse to p° when restricted to the essential image of p°. Since p is smooth, the shifted direct image functor $p_+[-n]$ is right adjoint to p° on the derived categories, where n is the dimension of the fibers of p. That is, for any $\mathscr{V} \in \mathcal{M}(\mathscr{D}_\lambda)$ there is a natural morphism of complexes

$$\mathrm{ad} : \mathscr{V} \to p_+ p^\circ \mathscr{V}[-n].$$

As p is a flat morphism between smooth projective varieties, the inverse image p° is exact on $\mathcal{M}(\mathscr{D}_\lambda)$. Recall that for surjective submersions the direct image functor p_+ is given by

$$\mathscr{V} \mapsto \mathbf{R}p_*(\mathscr{V} \otimes_{(\mathscr{D}^p_\lambda)^\circ} \mathcal{T}^\bullet_{X/X_\theta}(\mathscr{D}^p_\lambda)^\circ) = \mathbf{R}p_*(\mathscr{V} \otimes_{\mathscr{O}_X} \omega_{X/X_\theta} \otimes_{(\mathscr{D}^\circ_\lambda)^p} \mathcal{T}^\bullet_{X/X_\theta}(\mathscr{D}^\circ_\lambda)^p).$$

For all the reductions, we will use the first presentation of p_+, although the left $\mathscr{D}_\lambda$-module structure is obscured by this notation.

The relative tangent complex vanishes below the fiber dimension, which implies that for $\mathscr{V} \in \mathcal{M}(\mathscr{D}_\lambda)$ there is a standard truncation triangle

$$\begin{tikzcd}
 & \tau_{\geq 1}(p^\circ \mathscr{V} \otimes_{(\mathscr{D}^p_\lambda)^\circ} \mathcal{T}^\bullet_{X/X_\theta}(\mathscr{D}^p_\lambda)^\circ)[-n]) \arrow[dl, "{[1]}"] & \\
\mathrm{Ker}\, d^{-n} \arrow[rr] & & p^\circ \mathscr{V} \otimes_{(\mathscr{D}^p_\lambda)^\circ} \mathcal{T}^\bullet_{X/X_\theta}(\mathscr{D}^p_\lambda)^\circ[-n] \arrow[ul]
\end{tikzcd}$$

in the derived category, where $d^\bullet$ is the differential in $p^\circ \mathscr{V} \otimes_{(\mathscr{D}^p_\lambda)^\circ} \mathcal{T}^\bullet_{X/X_\theta}$. Since p_* is left exact, the long exact sequence obtained from applying $\mathbf{R}p_*$ to this triangle induces an isomorphism

$$p_* \mathrm{Ker}\, d^{-n} \simeq \mathrm{H}^{-n}(\mathbf{R}p_*(p^\circ \mathscr{V} \otimes_{(\mathscr{D}^p_\lambda)^\circ} \mathcal{T}^\bullet_{X/X_\theta}(\mathscr{D}^p_\lambda)^\circ))$$

in degree 0. The adjointness morphism thus descends to cohomology

$$\mathrm{H}^0(\mathrm{ad}) : \mathscr{V} \to \mathbf{R}^0 p_* \mathrm{Ker}\, d^{-n}.$$

In fact, since $p_+[-n]$ is left exact, the morphism $\mathrm{H}^0(\mathrm{ad})$ is injective.

The remainder of the proof proves surjectivity of $\mathrm{H}^0(\mathrm{ad})$. The projection formula for $\mathscr{O}$-modules produces the isomorphism

$$\mathbf{R}p_*(p^* \mathscr{V} \otimes \omega_{X/X_\theta} \otimes \mathcal{T}^\bullet_{X/X_\theta}) \simeq \mathscr{V} \otimes \mathbf{R}p_*(\omega_{X/X_\theta} \otimes \mathcal{T}^\bullet_{X/X_\theta});$$

it is also an isomorphism of left $\mathscr{D}_\lambda$-modules. To see this, we examine the tensor product $\omega_{X/X_\theta} \otimes \mathcal{T}^\bullet_{X/X_\theta}$. Define $F_x = p^{-1}(p(x))$ and let $\mathfrak{b}_x$ be the Borel corresponding to x and similarly $\mathfrak{p}_x$ the parabolic corresponding to $p(x)$. There is a short exact sequence

$$0 \to \mathfrak{p}_x/\mathfrak{b}_x \to \mathfrak{g}/\mathfrak{b}_x \to \mathfrak{g}/\mathfrak{p}_x \to 0$$

of the tangent spaces. From this sequence we obtain the isomorphisms

$$T_x\mathscr{T}_{X/X_\theta} = T_x\mathscr{T}_{F_x} \simeq \bar{\mathfrak{n}}_{\mathfrak{l},x} \text{ and } T_x\Omega_{X/X_\theta} = T_x\Omega_{F_x} \simeq \mathfrak{n}_{\mathfrak{l},x},$$

where $\mathfrak{n}_{\mathfrak{l},x}$ is the nilpotent subalgebra of a Levi factor $\mathfrak{l}_x$ of $\mathfrak{p}_x$ consisting of positive root spaces of type θ and $\bar{\mathfrak{n}}_{\mathfrak{l},x}$ is the opposite nilpotent subalgebra in $\mathfrak{l}_x$. In particular, the relative canonical sheaf of p is the homogeneous line bundle

$$\omega_{X/X_\theta} = \mathscr{O}(2\rho_\theta).$$

Furthermore, since $\mathcal{T}^{-n}_{X/X_\theta} = \wedge^n \mathscr{T}_{X/X_\theta} = \mathscr{O}(-2\rho_\theta)$, the tensor product $\omega_{X/X_\theta} \otimes \mathcal{T}^{-n}_{X/X_\theta} \simeq \mathscr{O}_X$ is p_*-acyclic. We claim additionally that for $k \neq n$ we have

$$\mathbf{R}p_*(\omega_{X/X_\theta} \otimes \mathcal{T}^k_{X/X_\theta}) \simeq 0. \tag{8}$$

Let $U \subset X_\theta$ be any open subset trivializing p. Then $\mathscr{T}^{-k}_{X/X_\theta}|_{U\times F}$ is isomorphic to $p_F^* \wedge^k \mathscr{T}_F$, where $p_F : U \times F \to F$ is the projection to $F \simeq F_x$. Similarly, $\omega_{X/X_\theta} = p_F^*\omega_F$. Then, we have

$$p_F^*(\omega_F \otimes \wedge^k \mathscr{T}_F)(U \otimes F) = \mathscr{O}_X(U) \otimes \Gamma(F, \omega_F \otimes \wedge^k \mathscr{T}_F).$$

To show (8) holds, it is enough to show $\Gamma(F, \omega_F \otimes \wedge^k \mathscr{T}_F)$ for all $k \neq n$.

Fix a Levi decomposition $B = HU$, let $\mathfrak{h}$ be the Lie algebra of H, and $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$ the maximal nilpotent subalgebra of $\mathfrak{b}$. Note that $\mathfrak{n}$ is the Lie algebra of the unipotent subgroup U. Let $\mathfrak{g}_\alpha$ be the root space in $\mathfrak{g}$ for root α, and Δ^+ the subset of positive roots. Let ρ, as usual, be the half-sum of positive roots.

The sheaf $\omega_X \otimes \wedge^k \mathscr{T}_X$ is the sheaf of sections of the vector bundle $G \times^B (\mathbb{C}_{2\rho} \otimes \wedge^k \bar{\mathfrak{n}})$, which has a filtration

$$F_p = G \times^B (\mathbb{C}_{2\rho} \otimes \wedge^k \bar{U}^p \bar{\mathfrak{n}}),$$

whose quotients F_p/F_{p+1} have as their sheaf of sections

$$\mathscr{O}(2\rho) \otimes \bigoplus_{\alpha \in \wedge^k \Delta_p^+} \mathscr{O}(-\alpha),$$

where $\wedge^k \Delta_p^+$ is the set of weights appearing in

$$\wedge^k(\bar{U}^p\bar{\mathfrak{n}}/\bar{U}^{p+1}\bar{\mathfrak{n}}).$$

That is, it is the set of k-fold sums of distinct length p positive roots. Therefore, for $\alpha \in \wedge^k \Delta_p^+$, the difference $2\rho - \alpha$ is a sum of positive roots, and hence not anti-dominant. By the Borel-Weil-Bott theorem, we have $\Gamma(X, \mathscr{O}(2\rho - \alpha)) = 0$. It follows that $\Gamma(X, \omega_X \otimes \wedge^k \mathscr{T}_X) = 0$ for all $k \neq n$.

The previous discussion implies that $p_+ p^\circ \mathscr{V}[-n] \simeq \mathscr{V}$. In fact, we proved something stronger:

$$p_+[-n]|_{p^\circ \mathcal{M}(\mathscr{D}_\lambda)} = p_*|_{p^\circ \mathcal{M}(\mathscr{D}_\lambda)}.$$

Additionally, the adjointness morphism $\mathrm{ad} : \mathscr{V} \to p_+ p^\circ \mathscr{V}[-n]$ is the identity. Therefore,

$$\mathrm{Hom}_{\mathscr{D}^p_\lambda}(p^\circ \mathscr{V}, p^\circ \mathscr{W}) = \mathrm{Hom}_{\mathscr{D}_\lambda}(\mathscr{V}, p_+ p^\circ \mathscr{W}[-n]) = \mathrm{Hom}_{\mathscr{D}_\lambda}(\mathscr{V}, \mathscr{W}).$$

Finally, we address the issue of commutativity with $\mathbf{R}\Gamma$. The equivalence $\mathscr{V} \simeq p_* p^\circ \mathscr{V}$ implies we have isomorphisms

$$\mathbf{R}\Gamma(X_\theta, \mathscr{V}) = \mathbf{R}\Gamma(X_\theta, p_* p^\circ \mathscr{V}) \simeq \mathbf{R}\Gamma(X, p^\circ \mathscr{V}),$$

of complexes of vector spaces. That the $\mathcal{D}_{[\lambda]}$-actions agree is a consequence of local triviality of the fibration $p : X \to X_\theta$. Locally, the tdo $\mathscr{D}^p_\lambda = \mathscr{D}_F \boxtimes \mathscr{D}_\lambda$ acts on $p^\circ \mathscr{V} = \mathscr{O}_F \boxtimes \mathscr{V}$ factor-wise. Therefore, the actions of $\mathcal{D}_{[\lambda]}$ and $\mathcal{U}^p_{[\lambda]}$ on $\mathbf{R}\Gamma(X_\theta, \mathscr{V})$ are related by p^*. □

Corollary 5.2. *The inverse image functor*

$$p^\circ : \mathrm{D}^{\mathrm{b}}(\mathscr{D}_\lambda, K) \to \mathrm{D}^{\mathrm{b}}(\mathscr{D}^p_\lambda, K)$$

is fully faithful for all λ, *and* $\mathbf{R}\Gamma \circ p^\circ = p^* \circ \mathbf{R}\Gamma$.

Proof. The projection p is K-equivariant. Moreover, the proof of the theorem lifts to $\mathcal{M}_w(\mathscr{D}_\lambda, K)$. We showed in §4.6 that the adjoint pair $p^\circ \dashv \mathbf{R}^{-n}p_+$ on $\mathcal{M}_w(\mathscr{D}_\lambda, K)$ defines an adjoint pair

$$\mathrm{D^b}(\mathscr{D}_\lambda, K) \underset{p_+[-n]}{\overset{p^\circ}{\rightleftarrows}} \mathrm{D^b}(\mathscr{D}^p_\lambda, K).$$

Since p° is fully faithful on $\mathcal{M}_w(\mathscr{D}_\lambda, K)$, the reduction principle implies it is also fully faithful on $\mathrm{D^b}(\mathscr{D}_\lambda, K)$.

The proof of the derived commutation property $\mathbf{R}\Gamma \circ p^\circ = p^* \circ \mathbf{R}\Gamma$ is the same as the proof for abelian categories. □

Corollary 5.3. *For $\lambda \in \mathfrak{h}^*_\theta$ anti-dominant, the functor $\Gamma : \mathcal{M}(\mathscr{D}_\lambda) \to \mathcal{M}(\mathcal{D}_\lambda)$ is exact. If λ is also regular, then Γ is also faithful.*

5.2. **Main Theorem.** Let X_θ be the partial flag variety of $\mathfrak{g}$ of type θ. Label the inclusion maps

$$\begin{tikzcd} x \arrow[r, "i_x"'] \arrow[rr, bend left, "j_x"] & Q \arrow[r, "i"'] & X_\theta. \end{tikzcd} \tag{9}$$

Let $p : X \to X_\theta$ be the natural fibration of the full flag variety X over X_θ. From (9) we obtain the following fiber products:

$$\begin{tikzcd} F_x \arrow[r, "i_x"'] \arrow[rr, bend left, "j_x"] \arrow[d, "p"] & F_Q \arrow[r, "i"'] \arrow[d, "p"] & X \arrow[d, "p"] \\ x \arrow[r, "i_x"] \arrow[rr, bend right, "j_x"'] & Q \arrow[r, "i"] & X_\theta. \end{tikzcd}$$

For a fixed point $x \in Q$, let $\mathfrak{p}_x$ be the parabolic subalgebra of $\mathfrak{g}$ corresponding to x, and let S_x be the stabilizer of x in K. Let $\mathfrak{n}_x$ be the nilpotent subalgebra of $\mathfrak{p}_x$. For a $(\mathfrak{p}_x, S_x)$-module Z, define $Z^\# := Z \otimes \wedge^{d_{X_\theta}} \bar{\mathfrak{n}}_x$ as a $(\mathfrak{p}_x, S_x)$-module via the adjoint action on the twisting factor. For a $(\mathfrak{p}_x, S_x)$-module V, the induced $(\mathfrak{g}, S_x)$-module is $\mathrm{ind}^{\mathfrak{g},S_x}_{\mathfrak{p}_x,S_x}(V) := \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p}_x)} V$.

Theorem 5.4 (Main Theorem 1.2)**.** *Let $\mathscr{D}_\lambda$ be a homogeneous tdo on X_θ and τ a connection on a K-orbit Q compatible with $\lambda + \rho_n$. Then there is a quasi-isomorphism*

$$\mathbf{R}\Gamma(X, p^\circ i_+\tau) \simeq \Gamma^{\mathrm{equi}}_{K,S_x}(\mathrm{ind}^{\mathfrak{g},S_x}_{\mathfrak{p}_x,S_x}(T_x\tau^\#))[d_Q]$$

in $\mathrm{D^b}(\mathcal{U}_{[\lambda-\rho_\theta]}, K)$. Upon taking cohomology, there is a convergent spectral sequence

$$\mathbf{R}^p\Gamma(X, p^\circ \mathbf{R}^q i_+\tau) \Longrightarrow \mathbf{R}^{p+q+d_Q}\Gamma_{K,S_x}(\mathrm{ind}^{\mathfrak{g},S_x}_{\mathfrak{p}_x,S_x}(T_x\tau^\#)). \tag{10}$$

Proof. The results of §4.6 define $\Gamma^{\mathrm{geo}}_{K,S_x}$ in the general derived equivariant setting and establish the isomorphism $i_+\tau \simeq \Gamma^{\mathrm{geo}}_{K,S_x} j_{x+}T_x\tau[d_Q]$ and commutativity properties ($\Gamma^{\mathrm{geo}}_{K,S_x} i_+ = i_+\Gamma_{K,S_x}$ and $\mathbf{R}\Gamma \circ \Gamma^{\mathrm{geo}}_{K,S_x} = \Gamma^{\mathrm{equi}}_{K,S_x} \circ \mathbf{R}\Gamma$), which culminate in the natural isomorphisms

$$\begin{aligned} \mathbf{R}\Gamma(X, p^\circ i_+(\text{ - })) &\simeq \mathbf{R}\Gamma(X, p^\circ \Gamma^{\mathrm{geo}}_{K,S_x} j_{x+}T_x(\text{ - }))[d_Q] \\ &\simeq \Gamma^{\mathrm{equi}}_{K,S_x}\mathbf{R}\Gamma(X, p^\circ j_{x+}T_x(\text{ - }))[d_Q]. \end{aligned} \tag{11}$$

Then by definition, we have $j_{x+}T_x\tau = j_{x*}(\mathscr{D}_{X_\theta \leftarrow x} \otimes T_x\tau)$, so

$$\begin{aligned} \mathbf{R}\Gamma(X, p^\circ i_+\tau) &\simeq \Gamma^{\mathrm{equi}}_{K,S_x}(\mathscr{D}_{X_\theta \leftarrow x} \otimes T_x\tau)[d_Q] \\ &= \Gamma^{\mathrm{equi}}_{K,S_x}(\mathcal{U}(\mathfrak{g})/\mathfrak{p}_x\mathcal{U}(\mathfrak{g}) \otimes T_x\omega^{-1}_{X_\theta} \otimes T_x\tau)[d_Q]. \end{aligned} \tag{12}$$

Note that $T_x\omega^{-1}_{X_\theta} = \wedge^{\mathrm{top}}\bar{\mathfrak{n}}_x$ and that the parabolic $\mathfrak{p}_x$ acts on $T_x\tau^\#$ by the F_x-invariant linear form $\lambda - \rho_n$. Therefore, there is an isomorphism

$$\mathcal{U}(\mathfrak{g})/\mathfrak{p}_x\mathcal{U}(\mathfrak{g}) \otimes T_x\tau^\# \simeq \mathrm{ind}^{\mathfrak{g},S_x}_{\mathfrak{p}_x,S_x}(T_x\tau^\#).$$

Finally, since i is a locally closed immersion and so the composition of the derived $\mathscr{O}_X$-module direct image $\mathbf{R}i_*$ with an exact functor. The spectral sequence is then seen to follow precisely from the Leray spectral sequence $\mathbf{R}^p\Gamma\mathbf{R}^q i_* \implies \mathbf{R}^{p+q}(\Gamma \circ i_*)$. □

When i_+ is exact, the left hand side of the spectral sequence collapses, and we have the following.

Corollary 5.5. *If i_+ is exact then the isomorphism $\mathbf{R}^p\Gamma(X, p^\circ i_+\tau) \simeq \mathbf{R}^{p+d_Q}\Gamma_{K,S_x}(\mathrm{ind}^{\mathfrak{g},S_x}_{\mathfrak{p}_x,S_x}(T_x\tau^\#))$ holds for all p.*

This is the case for any orbit Q in the full flag variety. Another example is for the open orbit in any partial flag variety of $\mathfrak{g}$ if the Cartan involution defining K is quasi-split. Alternatively, if we are working with twisted differential operators, but take λ to be anti-dominant, then Γ is exact, and we again find the left hand side collapses.

Corollary 5.6. *For λ anti-dominant, we have for all q an isomorphism*

$$\Gamma(X, p^\circ \mathbf{R}^q i_+\tau) \simeq \mathbf{R}^{q+d_Q}\Gamma_{K,S_x}(\mathrm{ind}^{\mathfrak{g},S_x}_{\mathfrak{p}_x,S_x}(T_x\tau^\#)).$$

Finally, we can combine the two corollaries to obtain a third:

Corollary 5.7. *For λ anti-dominant and i_+ exact, we have*

$$\Gamma(X, p^\circ i_+\tau) \simeq \mathbf{R}^{d_Q}\Gamma_{K,S_x}(\mathrm{ind}^{\mathfrak{g},S_x}_{\mathfrak{p}_x,S_x}(T_x\tau^\#))$$

and all other derived Zuckerman functors vanish.

Of possibly the greatest significance is the fact that the convergent spectral sequence (2) determines equalities in the Grothendieck group

$$[\mathbf{R}^{n+d_Q}\Gamma_{K,S_x}(\mathrm{ind}^{\mathfrak{g},S_x}_{\mathfrak{p}_x,S_x}(T_x\tau^\#))] = \sum_{p+q=n} [\mathbf{R}^p\Gamma(X, p^\circ\mathbf{R}^q i_+\tau)], \tag{13}$$

which may give additional geometric insight into the computation of composition series of degenerate principal series. Low rank examples of applications of this result are discussed in Chapter 8. of [7].

5.3. **Duality and Cohomologically Induced Modules.** Fix a Levi subgroup L_x of F_x. Then, $L_x \cap K$ is a maximal reductive subgroup of S_x. The representation $T_x\tau$ of the previous section is really a representation of $L_x \cap K$ extended trivially to S_x. For such representations V, we have equality of underlying $\mathfrak{g}$-modules

$$\mathrm{ind}^{\mathfrak{g},S_x}_{\mathfrak{p}_x,S_x}(V) = \mathrm{ind}^{\mathfrak{g},L_x}_{\mathfrak{p},L_x}(V).$$

Abstractly, let $\mathfrak{p} \subset \mathfrak{g}$ be any parabolic, and let $L \subset K$ be a reductive subgroup such that $\mathfrak{l} \subset \mathfrak{p}$. Then, the left adjoint to the forgetful functor from $(\mathfrak{g}, L)$-modules to $(\mathfrak{p}, L)$-modules is $\mathrm{ind}^{\mathfrak{g},L}_{\mathfrak{p},L}$ and the right adjoint is

$$\mathrm{pro}^{\mathfrak{g},L}_{\mathfrak{p},L}(\ \text{-}\ ) = \mathrm{Hom}_{\mathfrak{p}}(\mathcal{U}(\mathfrak{g}),\ \text{-}\ )_{[L]},$$

where the $[L]$ indicates that we take L-finite vectors. For any $(\mathfrak{g}, K)$-module V, define the contragredient $V^\vee = V^*_{[K]}$. Then, we have the following lemma.

Lemma 5.8 ([5], Lemma 3.1)**.** *For V any $(\mathfrak{p}, L)$-module,*

$$\mathrm{ind}^{\mathfrak{g},L}_{\mathfrak{p},L}(V)^\vee = \mathrm{pro}^{\mathfrak{g},L}_{\mathfrak{p},L}(V^\vee).$$

To identify the modules of Theorem 1.2. as contragredient to cohomologically induced modules, we need to introduce Zuckerman duality:

Theorem 5.9. *Let G, P, and L be as above and let V be a $(\mathfrak{l}, L\cap K)$-module. Let $\mathfrak{n}$ be the nilradical of $\mathfrak{p}$ and let $\mathfrak{o} = \mathfrak{k}\cap\mathfrak{n} \oplus \mathfrak{k}\cap\bar{\mathfrak{n}}$ and $s = \dim \mathfrak{k}\cap\mathfrak{n}$. Then, for all $i \geq 0$, there is an isomorphism of $(\mathfrak{g}, K)$-modules*

$$\Gamma^i_{K,L\cap K}(V^\vee) \simeq \Gamma^{2s-i}_{K,L\cap K}(\wedge^{\mathrm{top}}\mathfrak{o} \otimes V)^\vee.$$

See [6, Cor. 6.1.9] for the proof of this theorem in the case of real groups. The one-dimensional $(\mathfrak{l}, L\cap K)$-module $\wedge^{\mathrm{top}}\mathfrak{o}$ is trivial for $\mathfrak{l}$, but may have a non-trivial action of the component group of $L \cap K$. Additionally, we have

Theorem 5.10 ([12], Theorem 1.13)**.** *Let $S \subset K$ be a subgroup and T its Levi factor. The Zuckerman functor $\mathbf{R}\Gamma_{K,S}$ is the restriction of $\mathbf{R}\Gamma_{K,T}$ to $\mathrm{D}(\mathcal{M}(\mathfrak{g}, S))$.*

For any $(\mathfrak{p}_x, L_x \cap K)$-module V, let $V^{\sim} = V \otimes \wedge^{\mathrm{top}}\mathfrak{n}_x$. Then, we have $(V^{\#})^{\vee} = (V^{\vee})^{\sim}$. The ith cohomologically induced module of V is defined to be

$$\mathcal{R}^i(V) = \mathbf{R}^i\Gamma_{K,L_x\cap K}(\mathrm{pro}^{\mathfrak{g},L_x\cap K}_{\bar{\mathfrak{p}}_x,L_x\cap K}(V^{\sim})).$$

Properties can be found in [6] or [8]. As a consequence of Theorem 1.2, we have the following corollary.

Corollary 5.11. *With the same hypotheses as Theorem 1.2, let $\mathfrak{o} = \mathfrak{k} \cap \mathfrak{n}_x \oplus \mathfrak{k} \cap \bar{\mathfrak{n}}_x$. Then*

$$\mathbf{R}^{d_Q+i}_{K,S_x}(\mathrm{ind}^{\mathfrak{g},L_x\cap K}_{\mathfrak{p}_x,L_x\cap K}(T_x\tau^{\#}))^{\vee} \simeq \mathcal{R}^{d_Q-i}((T_x\tau^{\vee} \otimes \wedge^{2d_Q}\mathfrak{o}^{\vee})^{\sim}).$$

Proof. The duality results yield the isomorphism

$$\mathbf{R}^{d_Q+i}_{K,S_x}(\mathrm{ind}^{\mathfrak{g},L_x\cap K}_{\mathfrak{p}_x,L_x\cap K}(T_x\tau^{\#}))^{\vee} \simeq \mathbf{R}^{d_Q-i}\Gamma_{K,S_x}(\mathrm{pro}^{\mathfrak{g},L_x\cap K}_{\bar{\mathfrak{p}}_x,L_x\cap K}((T_x\tau^{\#} \otimes \wedge^{2d_Q}\mathfrak{o})^{\vee})).$$

The observation that $(V^{\#})^{\vee} = (V^{\vee})^{\sim}$ for any $(\mathfrak{p}_x, L_x \cap K)$-module implies

$$(T_x\tau^{\#} \otimes \wedge^{2d_Q}\mathfrak{o})^{\vee} = (T_x\tau^{\vee} \otimes \wedge^{2d_Q}\mathfrak{o}^{\vee})^{\sim},$$

and Theorem 5.10. completes the proof. □

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